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Thermodynamics of asymptotically locally AdS spacetimes

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Abstract: We formulate the variational problem for AdS gravity with Dirichlet boundary conditions and demonstrate that the covariant counterterms are necessary to make the variational problem well-posed. The holographic charges associated with asymptotic symmetries are then rederived via Noether's theorem and 'covariant phase space' techniques. This allows us to prove the first law of black hole mechanics for general asymptotically locally AdS black hole spacetimes. We illustrate our discussion by computing the conserved charges and verifying the first law for the four dimensional Kerr-Newman-AdS and the five dimensional Kerr-AdS black holes.

Keywords: AdS-CFT and dS-CFT Correspondence, Black Holes in String Theory, Black Holes.

Contents

1. Introduction

The study of gravitational theories with negative cosmological constant has been an active area of research. The recent interest stems in part from the duality between asymptotically AdS spacetimes and quantum field theories residing at the conformal boundary of the spacetime. One of the implications of the AdS/CFT duality is that such gravitational theories should exist under more general boundary conditions than those considered in the past. The boundary conditions that were considered in the previous works, see for instance [1, 2], are such that the spacetime asymptotically approaches the exact AdS solution. In the AdS/CFT correspondence the fields parameterizing the boundary conditions of bulk fields are interpreted as sources that couple to gauge invariant operators, and since such sources are in general arbitrary, we are led to consider spacetimes with general boundary conditions. In particular, instead of considering the boundary conformal structure to be that of exact AdS, one can consider a general conformal structure. Such more general boundary conditions have been considered in the mathematics literature [3] (see [4, 5] for reviews.) We will call such spacetimes asymptotically locally AdS (AlAdS) spacetimes.

An important aspect of asymptotically AdS spacetimes, which has attracted considerable attention over the years, is the definition of conserved charges associated with the asymptotic symmetries of such spacetimes [6, 1, 2, 7, 8], see also [9] and references therein. Besides the difficulty of defining precisely what one means by 'asymptotically AdS spacetimes', the main obstruction in defining such conserved charges is the fact that the non-compactness of these spacetimes causes various 'natural candidates' for conserved quantities, such as Komar integrals, to diverge [10]. One is then forced to introduce some regularization procedure, which is inherently ambiguous.

To circumvent this difficulty, most approaches either require that the spacetime approaches asymptotically the exact AdS metric and they then use the special properties of AdS to construct conserved quantities which are manifestly finite, e.g. [1, 2], or they embed the spacetime into a spacetime with the same asymptotics and then define manifestly finite conserved quantities relative to the ambient spacetime, e.g. [11, 12]. Although the philosophy and the precise definition of the conserved charges varies among these methods, they all implement some form of 'background subtraction'. Therefore, despite the simplicity and elegance of some of these methods, they are all ultimately rather restrictive since not all asymptotically locally AdS spacetimes can be embedded in a suitable ambient spacetime.

Inspired by the AdS/CFT correspondence [13 – 15], a general background independent definition of the conserved charges for any AlAdS spacetime was developed in [16 – 19]. In a first step one associates to any AlAdS spacetime a finite Brown-York stress energy tensor [20] obtained by varying the on-shell gravitational action w.r.t. the boundary metric. Finite conserved charges associated to asymptotic symmetries are then obtained from this holographic stress energy tensor by a standard procedure. The finiteness of the on-shell action and thus of the holographic stress energy tensor and of all holographic conserved charges is achieved by adding to the gravitational action a set of boundary covariant counterterms. This method is thus also referred to as the 'method of covariant counterterms'. The method has been used extensively over the last few years. Nevertheless, there exists an ongoing debate about the connection between these holographic charges and the various alternative definitions of conserved charges and, in particular, regarding the validity of the first law of black hole mechanics [21]. One of the aims of this paper to clarify the concept of the holographic charges (see also [22]) and, in particular, to prove in general that *all* AlAdS black holes satisfy the first law of black hole mechanics and the charges entering this law are the holographic charges.

The paper is organized as follows. In the next section we review the definition of asymptotically locally AdS spacetimes. In section 3, after reviewing the asymptotic analysis, we formulate the variational problem with Dirichlet boundary conditions for AdS gravity and we demonstrate the need for the counterterms. In section 4 we derive an alternative expression for the holographic charges using Noether's theorem and we show that these are reproduced by the covariant phase space method of Wald et al. [23, 24]. We then use these results in section 5 to prove the first law of black hole mechanics for any AlAdS spacetime. We conclude with two examples in section 6, namely the four dimensional Kerr-Newman-AdS black hole and the five dimensional Kerr-AdS black hole, which provides an illustration of the role of the conformal anomaly. Various technical results are collected in the appendices. In appendix E we comment on the connection between the 'conformal mass' of Ashtekar and Magnon [1] and the holographic mass.

2. Asymptotically locally AdS spacetimes

We briefly discuss in this section the definition of asymptotically locally AdS spacetimes. For more details we refer to [25][1] (see also the math reviews [4, 5]).

Recall that AdS_{d+1} is the maximally symmetric solution of Einstein's equations with negative cosmological constant, $\Lambda = -d(d-1)/2l^2$, where l is the radius of AdS_{d+1} (we set $l = 1$ from now on; one can easily reinstate this factor in all equations by dimensional analysis). Its curvature tensor is given by

$$R_{\mu\nu\kappa\lambda} = g_{\mu\lambda}g_{\nu\kappa} - g_{\kappa\mu}g_{\nu\lambda}\,, \tag{2.1}$$

and it has a conformal boundary with topology $\mathbb{R} \times \mathbf{S}^{d-1}$, where $\mathbb{R}$ corresponds to the time direction.[2] Asymptotically locally AdS (AlAdS) spacetimes are solutions of Einstein's equations with Riemann tensor approaching (2.1) asymptotically (in a sense to be specified shortly). A more restrictive class of spacetimes are the ones which asymptotically become exactly AdS spacetimes; these were called Asymptotically AdS (AAdS) in [1, 2]. One can easily specialize our results to AAdS spacetimes and we shall do so in order to compare with existing literature.

The spacetimes $\mathcal{M}$ we consider are in particular *conformally compact manifolds* [26]. This means that $\mathcal{M}$ is the interior of a manifold-with-boundary $\overline{\mathcal{M}}$, and the metric g has

[1]Note that the asymptotically locally AdS spacetimes here were called asymptotically AdS spacetimes in [25].

[2]Strictly speaking, the time coordinate is compact but one can go to the universal cover $CAdS$ where the time coordinate is unfolded. In this paper the time coordinate is always considered non-compact (except when we discuss the euclidean continuation).

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a second order pole at the boundary $B = \partial\mathcal{M}$, but there exists a defining function (i.e. $z(B) = 0$, $dz(B) \neq 0$ and $z(\mathcal{M}) > 0$) such that[3]

$$\mathtt{g} = z^2 g \tag{2.2}$$

smoothly extends to $\overline{\mathcal{M}}$, $\mathtt{g}|_B = \mathtt{g}_{(0)}$, and is non-degenerate. A standard argument (see for instance [15]) implies that the boundary B is equipped with a conformal class of metrics and $\mathtt{g}_{(0)}$ is a representative of the conformal class. A conformally compact manifold that is also Einstein (i.e. solves Einstein's equations) is by definition an asymptotically locally AdS spacetime.

The most general asymptotics of such spacetimes was determined in [3] for pure gravity and their analysis extends straightforwardly to include matter with soft enough behavior at infinity, see for instance [18, 27 – 29]. Near the boundary, one can always choose coordinates in which the metric takes the form,[4]

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \frac{dz^2}{z^2} + \frac{1}{z^2} \mathtt{g}_{ij}(x,z) dx^i dx^j \,, \\ \mathtt{g}(x,z) &= \mathtt{g}_{(0)} + z\mathtt{g}_{(1)} \cdots + z^d \mathtt{g}_{(d)} + \mathtt{h}_{(d)} z^d \log z^2 + \cdots \,. \end{aligned} \tag{2.3}$$

In these coordinates the conformal boundary is located at $z = 0$ and $\mathtt{g}_{(0)}$ is a representative of the conformal structure. The asymptotic analysis reveals that all coefficients shown above except the traceless and divergenceless part of $\mathtt{g}_{(d)}$ are locally determined in terms of boundary data. The logarithmic term appears only in even (boundary) dimensions (for pure gravity; if matter fields are included, then a logarithmic term can appear in odd dimensions as well [18]) and is proportional [18] to the metric variation of the integrated holographic conformal anomaly [16]. A simple computation shows that the Riemann tensor of (2.3) is of the form (2.1) up to a correction of order z. This continues to be true in the presence of matter if the dominant contribution of their stress energy tensor as we approach the boundary comes from the cosmological constant. This is true for matter that corresponds to marginal or relevant operators of the dual theory in the AdS/CFT duality.

A very useful reformulation of the asymptotic analysis can be achieved by observing that for AlAdS spacetimes the radial derivative is to leading order equal to the dilatation operator [29, 30]. Let us first write the metric (2.3) in the form

$$ds^2 = dr^2 + \gamma_{ij}(x,r) dx^i dx^j \,, \tag{2.4}$$

where $z = \exp(-r)$. The statement is that

$$\partial_r = \delta_D + \mathcal{O}(e^{-r}) \,, \tag{2.5}$$

with

$$\delta_D = \int d^d x \left(2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} + (\Delta_I - d)\Phi^I \frac{\delta}{\delta\Phi^I} + \cdots \right) , \tag{2.6}$$

[3] Note that in [25] the symbol g was used for the unphysical metric, but here we use instead $\mathtt{g}$ and use g to denote the bulk metric.

[4] In most examples in the literature the odd coefficients $\mathtt{g}_{(2k+1)}$ vanish (except when $2k+1 = d$, the boundary dimension). In such cases, it is more convenient [16] to use instead of z a new radial coordinate $\rho = z^2$.

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where we have included the contribution of massive scalars Φ^I dual to operators of dimension Δ_I and the dots indicate the contribution of other matter fields. The asymptotic analysis can now be very effectively performed [29] by expanding all objects in eigenfunctions of the dilatation operator and organizing the terms in the field equations according to their dilatation weight.

Let us now see how AdS_{d+1} and $AAdS_{d+1}$ spacetimes fit in this framework. AdS_{d+1} is conformally flat and this implies [31] that $\mathsf{g}_{(0)}$ is conformally flat as well and the expansion (2.3) terminates at order z^4,

$$\mathsf{g}_{(4)} = \frac{1}{4}(\mathsf{g}_{(2)})^2\,, \qquad \mathsf{g}_{(2)ij} = -\frac{1}{d-2}(R_{ij} - \frac{1}{2(d-1)}R\mathsf{g}_{(0)ij})\,, \tag{2.7}$$

where R_{ij} is the Ricci tensor of $\mathsf{g}_{(0)}$ ($d = 2$ is a special case, see [31] for the expression of $\mathsf{g}_{(2)}$) and $\mathsf{g}_{(0)}$ may be chosen to be the standard metric on $\mathbb{R} \times \mathbf{S}^{d-1}$. Any $AAdS_{d+1}$ spacetime has a conformally flat representative $\mathsf{g}_{(0)}$ as well, which implies that all coefficients up to $\mathsf{g}_{(d)}$ are the same[5] as for AdS_{d+1}, but $\mathsf{g}_{(d)}$ is different. In both cases, the logarithmic term is absent.

AlAdS spacetimes have an arbitrary conformal structure $[\mathsf{g}_{(0)}]$ and a general $\mathsf{g}_{(d)}$, the logarithmic term is in general non-vanishing, and there is no *a priori* restriction on the topology of the conformal boundary. The mathematical structure of these spacetimes (or their euclidean counterparts) is under current investigation in the mathematics community, see [5] and references therein. For instance, it is has not yet been established how many, if any, global solutions exist given a conformal structure, although given (sufficiently regular) $\mathsf{g}_{(0)}$ and $\mathsf{g}_{(d)}$ a unique solution exists in a thickening $B \times [0, \epsilon)$ of the boundary B. On the other hand, interesting examples of such spacetimes exist, see [5] for a collection of examples.

There is an important difference between even and odd dimensions. When the spacetime is odd dimensional, there is a conformally invariant quantity $A[\mathsf{g}_{(0)}]$ one can construct using the boundary conformal structure $[\mathsf{g}_{(0)}]$, namely the integral of the holographic conformal anomaly [16] (called renormalized volume in the math literature [4]).[6] The conformal anomaly was found in [16] by considering the response of the *renormalized* on-shell action to Weyl transformations: in order to render finite the on-shell gravitational action (which diverges due to the infinite volume of the AlAdS spacetime) one is forced to add a certain number of boundary counterterms and the latter induce an anomalous Weyl transformation. We will soon rediscover the need for counterterms and the anomaly via a different argument. For now we note that classically, the bulk metric determines a conformal structure $[\mathsf{g}_{(0)}]$, and in odd dimensions, the latter determines a conformal invariant, the integrated anomaly $A[\mathsf{g}_{(0)}]$.

In the next section we investigate the variational problem for AlAdS spacetimes. Given that a bulk metric is associated with a conformal structure at infinity, we would like to

[5]In the presence of matter however these coefficients could acquire matter field dependence, see [28] for an example.

[6]When certain matter fields are present one has additional conformal invariants in all dimensions, namely the matter conformal anomalies [32]; these play a similar role to the gravitational conformal anomaly.

formulate the variational problem such that the conformal structure is kept fixed. We will show that this is indeed possible when the anomaly vanishes. When the anomaly is non-vanishing, however, the variational problem is more subtle: instead of keeping fixed a conformal class, one chooses a representative and arranges such that the dependence of the theory on different representatives is determined by the conformal class via the conformal anomaly. In other words, we need to choose a representative in order to define the theory but the difference between different choices is governed by the conformal class. We will see that in all cases the variational problem requires new boundary terms and these are precisely the counterterms!

3. Counterterms and the variational problem for AdS gravity

3.1 The theory

We will consider in this section the variational problem for AdS gravity coupled to scalars and a Maxwell field. Other matter fields, like forms and non-abelian gauge fields, can be easily incorporated in the analysis, but for simplicity we do not include them. Moreover, to keep the analysis general we do not include any Chern-Simons terms since their particular form depends on the spacetime dimension. Within this framework we consider the most general lagrangian consistent with the fact that the field equations admit a solution that is asymptotically locally AdS. The lagrangian D-form ($D{=}d+1$) is given by

$$\mathbf{L} = \left(\frac{1}{2\kappa^2}R - V(\Phi)\right) * \mathbf{1} - \frac{1}{2}G_{IJ}(\Phi)\mathrm{d}\Phi^I \wedge *\mathrm{d}\Phi^J - \frac{1}{2}U(\Phi)\mathbf{F}\wedge *\mathbf{F}\,, \tag{3.1}$$

where we use mostly plus signature and $\mathbf{F} = \mathrm{d}\mathbf{A}$ and $V(\Phi)$, $U(\Phi)$ and $G_{IJ}(\Phi)$ are only constrained by the requirement that the field equations admit AlAdS solutions. The exact conditions follow from the asymptotic analysis discussed in the next subsection, but we will not need the detailed form of the conditions in this paper.

The variation of the lagrangian with respect to arbitrary field variations takes the form

$$\delta\mathbf{L} = \mathbf{E}\delta\psi + \mathrm{d}\mathbf{\Theta}(\psi,\delta\psi)\,, \tag{3.2}$$

where we use $\psi = (g_{\mu\nu}, A_\mu, \Phi^I)$ to denote collectively all fields and $\mathbf{E}$ is the equations of motion D-form. More specifically, we have

$$\delta\mathbf{L} = \mathbf{E}^{\mu\nu}{}_{(1)}\delta g_{\mu\nu} + \mathbf{E}^{\mu}{}_{(2)}\delta A_\mu + \mathbf{E}^{(3)}_I\delta\Phi^I + \mathrm{d}\mathbf{\Theta}(\psi,\delta\psi)\,, \tag{3.3}$$

where

$$\begin{aligned}
\mathbf{E}_{(1)}{}^{\mu\nu} &= -\frac{1}{2\kappa^2}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \kappa^2\tilde{T}^{\mu\nu}\right) * \mathbf{1}\,,\\
\mathbf{E}_{(2)}{}^{\nu} &= \nabla_\mu(U(\Phi)F^{\mu\nu}) * \mathbf{1}\,,\\
\mathbf{E}^{(3)}_I &= \left(\nabla^\mu(G_{IJ}(\Phi)\partial_\mu\Phi^J) - \frac{1}{2}\frac{\partial G_{JK}}{\partial\Phi^I}\partial_\mu\Phi^J\partial^\mu\Phi^K - \frac{\partial V}{\partial\Phi^I} - \frac{1}{4}\frac{\partial U}{\partial\Phi^I}F_{\mu\nu}F^{\mu\nu}\right) * \mathbf{1}\,,
\end{aligned} \tag{3.4}$$

and the matter stress tensor is given by

$$\tilde{T}_{\mu\nu} = G_{IJ}(\Phi)\partial_\mu\Phi^I\partial_\nu\Phi^J + U(\Phi)F_{\mu\rho}F_\nu{}^\rho - g_{\mu\nu}\mathcal{L}_{\mathrm{m}}\,, \tag{3.5}$$

with $\mathcal{L}_{\rm m}$ denoting the matter part of the lagrangian. Moreover,

$$\mathbf{\Theta}(\psi, \delta\psi) = - * v(\psi, \delta\psi)\,, \tag{3.6}$$

where

$$v^\mu = -\frac{1}{2\kappa^2}(g^{\mu\rho}\nabla^\sigma \delta g_{\rho\sigma} - g^{\rho\sigma}\nabla^\mu \delta g_{\rho\sigma}) + G_{IJ}(\Phi)\delta\Phi^I \nabla^\mu \Phi^J + U(\Phi)F^{\mu\nu}\delta A_\nu\,. \tag{3.7}$$

3.2 Asymptotic analysis

In this section we discuss the asymptotic solutions to the field equations. To formulate the problem we use a radial coordinate r emanating orthogonally from the boundary in order to foliate spacetime into timelike hypersurfaces Σ_r diffeomorphic to the conformal boundary $\partial\mathcal{M}$. This can always be done at least in the vicinity of the boundary. We then regulate the theory by introducing a cut-off hypersurface Σ_{r_o}.

The most convenient way to perform the asymptotic analysis is by using a 'radial hamiltonian analysis' where the radial coordinate plays the role of time [29] (see [33–35] for earlier work). In this formalism, one uses the Gauss-Codazzi equations to express the bulk equations of motion in terms of quantities intrinsic to the radial hypersurfaces Σ_r. In the gauge

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j\,, \qquad A_r = 0\,, \tag{3.8}$$

the resulting equations of motion are given in appendix A and can be viewed as Hamilton's equations for the 'radial canonical momenta',

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}(K^{ij} - K\gamma^{ij})\,, \qquad \pi^i = -\sqrt{-\gamma}U(\Phi)\dot{A}^i\,, \qquad \pi^I = -\sqrt{-\gamma}G_{IJ}(\Phi)\dot{\Phi}^I\,, \tag{3.9}$$

where $K_{ij} = \frac{1}{2}\dot{\gamma}_{ij}$ is the second fundamental form of the hypersurfaces Σ_r and the dot denotes differentiation w.r.t. the radial coordinate.

Within this framework, one is also able to express the on-shell value of the regulated action as an integral over the surface Σ_{r_o} by introducing a Σ_r-covariant variable λ such that

$$\int_{\mathcal{M}_{r_o}} \mathbf{L}_{\rm on-shell} = \int_{\mathcal{M}_{r_o}} d^{d+1}x\sqrt{-g}\left(\mathcal{L}_{\rm m} - \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) \equiv -\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\lambda\,. \tag{3.10}$$

Since Σ_{r_o} is compact, λ is only defined up to a total divergence. We will soon fix this ambiguity by making a choice that simplifies the analysis. Taking the radial derivative of both sides of (3.10) we deduce that λ must satisfy the differential equation

$$\dot{\lambda} + K\lambda + \kappa^2\left(\mathcal{L}_{\rm m} - \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) = 0\,. \tag{3.11}$$

The regulated on-shell action (with the Gibbons-Hawking term included) is then given by

$$I_{r_o} = \frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}(K - \lambda)\,. \tag{3.12}$$

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The radial momenta are now related to the on-shell action (see for instance [29] and (3.42) below) by

$$\pi^{ij} = \frac{\delta I_{r_o}}{\delta \gamma_{ij}}\,, \qquad \pi^i = \frac{\delta I_{r_o}}{\delta A_i}\,, \qquad \pi_I = \frac{\delta I_{r_o}}{\delta \Phi^I}\,. \tag{3.13}$$

These expressions can be utilized to fix the total divergence ambiguity in λ. In particular, since I_{r_o} is unaffected by the addition of a total divergence to λ, so are the momenta. We now argue that by adding an appropriate total derivative term to λ we can always ensure that the identity

$$\pi^{ij}\delta\gamma_{ij} + \pi^i \delta A_i + \pi_I \delta\Phi^I = \frac{1}{\kappa^2}\delta\left[\sqrt{-\gamma}(K-\lambda)\right] \tag{3.14}$$

holds *without* the integral over Σ_{r_o}. This can always be achieved by the following procedure.[7] Take first any λ satisfying the definition (3.10). The variation $\delta\left[\sqrt{-\gamma}(K-\lambda)\right]$ will then generically produce terms with derivatives acting on the variations of the induced fields $\delta\gamma_{ij}$, δA_i and $\delta\Phi^I$. These derivatives can be moved to the coefficients of the field variations by integration by parts. When all derivatives acting on the field variations are removed, (3.13) guarantees that the coefficients of the field variations are precisely the radial momenta. Now, the total derivative terms which are produced by this procedure can be absorbed into λ. In writing (3.14), we assume that such a procedure has been performed.

To carry out the asymptotic analysis we use the fact that asymptotically the radial derivative becomes equal to the dilatation operator, which with the current field content takes the form

$$\delta_D = \int d^d x \left(2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} + \sum_I (\Delta_I - d)\Phi^I \frac{\delta}{\delta\Phi^I}\right). \tag{3.15}$$

This follows from the fact that on-shell, one can identify the radial derivative with the functional differential operator

$$\partial_r = \int d^d x \left(2K_{ij}[\gamma, A, \Phi]\frac{\delta}{\delta\gamma_{ij}} + \dot{A}_i[\gamma, A, \Phi]\frac{\delta}{\delta A_i} + \dot{\Phi}^I[\gamma, A, \Phi]\frac{\delta}{\delta\Phi^I}\right) = \delta_D + \mathcal{O}(e^{-r})\,, \tag{3.16}$$

where the asymptotic behavior of the fields has been used. This observation motivates an expansion of the momenta and the on-shell action in eigenfunctions of δ_D:

$$\begin{aligned}
\pi^i_j &= \sqrt{-\gamma}\left(\pi_{(0)}{}^i_j + \pi_{(2)}{}^i_j + \cdots + \pi_{(d)}{}^i_j + \tilde{\pi}_{(d)}{}^i_j \log \mathrm{e}^{-2r} + \cdots\right),\\
\pi^i &= \sqrt{-\gamma}\left(\pi_{(3)}{}^i + \pi_{(4)}{}^i + \cdots + \pi_{(d)}{}^i + \tilde{\pi}_{(d)}{}^i \log \mathrm{e}^{-2r} + \cdots\right),\\
\pi_I &= \sqrt{-\gamma}\Bigg(\sum_{d-\Delta_I \le s < \Delta_I} \pi_{(s)I} + \pi_{(\Delta_I)} + \tilde{\pi}_{(\Delta_I)I} \log \mathrm{e}^{-2r} + \cdots \Bigg),\\
\lambda &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log \mathrm{e}^{-2r} + \cdots .
\end{aligned} \tag{3.17}$$

[7]This argument holds only for the *local* part of λ and not for the non-local part $\lambda_{(d)}$ (see (3.17) below for the definition of this term) which satisfies only the integrated version of (3.14). However, for the special case of dilatations, $\delta = \delta_D$, $\lambda_{(d)}$ does satisfy (3.14). To see this consider an infinitesimal Weyl transformation of the renormalized action (3.23): $\delta_\sigma I_{\rm ren} = -\frac{2}{\kappa^2}\int_{\partial\mathcal{M}}\sqrt{-\gamma}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)})\delta\sigma$. But from the renormalized version of (3.13) we also have: $\delta_\sigma I_{\rm ren} = \int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\left[2\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\Phi^I\right]\delta\sigma$. Since $\delta\sigma$ is arbitrary, we can equate the integrands, which gives the same result as (3.14) when specialized to δ_D.

All terms in these expansions transform under dilatations according to their subscript, e.g. $\pi_{(n)}$ transforms as

$$\delta_D \pi_{(n)} = -n\pi_{(n)}\,, \tag{3.18}$$

except for the normalizable ("vev") part of the expansions, $\pi_{(d)}{}^i_j$, $\pi_{(d)}{}^i$, $\pi_{(\Delta_I)}$, $\lambda_{(d)}$, which transform inhomogeneously, with the inhomogeneous term equal to (-2) the coefficient of the logarithmic piece. For example,

$$\delta_D \pi_{(d)}{}^i_j = -d\pi_{(d)}{}^i_j - 2\tilde{\pi}_{(d)}{}^i_j\,. \tag{3.19}$$

(Note that the transformation of the volume element, namely $\delta_D\sqrt{-\gamma} = d\sqrt{-\gamma}$, should be taken into account).

The significance of writing the radial derivative in the form (3.16) is that the above expansions imply that the radial derivative can also be expanded in a series of covariant functional operators, $\delta_{(n)}$, of successively higher dilatation weight that commute with the dilatation operator. Namely,

$$\partial_r = \delta_D + \delta_{(1)} + \cdots\,. \tag{3.20}$$

This allows us to perform the asymptotic analysis in a covariant way by substituting the expansions (3.17) and the expansion for the radial derivative in the field equations and collecting terms with the same weight. This determines uniquely and *locally* all coefficients of momenta in (3.17), except for the traceless and divergenceless part of $\pi_{(d)}{}^i_j$ and the divergenceless part of $\pi_{(d)}{}^i$, in terms of the boundary data, i.e. in terms of the induced fields on Σ_{r_o}.

Ward identities. The divergence of $\pi_{(d)}{}^i_j$ and $\pi_{(d)}{}^i$, which are determined respectively by the second equation in (A.2) and the first equation in (A.4), yield the Ward identities related to boundary diffeomorphisms and U(1) gauge transformations respectively. These read,

$$\begin{aligned} 2D_i\pi_{(d)}{}^i_j + \pi_{(d)}{}^i F_{ij} - \pi_I \partial_j \Phi^I &= 0\,, \\ D_i \pi_{(d)}{}^i &= 0\,. \end{aligned} \tag{3.21}$$

The trace Ward identity follows from the explicit expression of $\pi_{(d)}{}^i_i$ obtained by solving asymptotically the field equations. Alternatively, one can follow the argument in footnote 7. As explained, the result is identical to that obtained by specializing (3.14) to dilatations and considering the terms of weight d - although only the integrated version of (3.14) holds for these terms. The result is

$$2\pi_{(d)}{}^i_i + \sum_I (\Delta_I - d)\pi_{(\Delta_I)I}\Phi^I = -\frac{2}{\kappa^2}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}) \equiv \mathcal{A}\,, \tag{3.22}$$

where $\mathcal{A}$ is the trace anomaly [16].

Renormalized action. Finally, the renormalized action is defined as[8]

$$I_{\rm ren}=\lim_{r_o\to\infty}(I_{r_o}+I_{\rm ct})=\frac{1}{\kappa^2}\int_{\partial\mathcal{M}}d^dx\sqrt{-\gamma}(K_{(d)}-\lambda_{(d)})\,, \tag{3.23}$$

where the counterterm action, $I_{\rm ct}$, is given by

$$I_{\rm ct}=-\frac{1}{\kappa^2}\int_{\Sigma_{r_o}}d^dx\sqrt{-\gamma}\left(\sum_{n=0}^{d-1}(K_{(n)}-\lambda_{(n)})+(\tilde{K}_{(d)}-\tilde{\lambda}_{(d)})\log e^{-2r_o}\right). \tag{3.24}$$

3.3 Gauge invariance of the renormalized action

In this section we first determine the most general bulk diffeomorphisms and U(1) gauge transformations which preserve the gauge (3.8). We note that this gauge need only be preserved up to terms of next-to-normalizable mode order, i.e. up to order $e^{-(d-1)r}$. Such transformations leave invariant the functional form of the boundary conditions, of the asymptotic solutions, and of the counterterm action on the regulated boundary Σ_{r_o}. Subsequently, we derive the maximal subset of gauge-preserving transformations that leave the renormalized action invariant, where only the functional form of the boundary conditions is imposed, namely we require that

$$\gamma_{ij}(x,r)\sim e^{2r}\mathtt{g}_{(0)ij}(x)\,,\qquad A_i(x,r)\sim\mathtt{A}_{(0)i}(x)\,,\qquad \Phi^I(x,r)\sim\phi^I_{(0)}(x)e^{-(d-\Delta_I)r}\,, \tag{3.25}$$

but no conditions are imposed on $\mathtt{g}_{(0)ij}$, $\mathtt{A}_{(0)i}$, $\phi^I_{(0)}$. Notice that the transformations below do act on these coefficients.

In the gauge (3.8), the Lie derivative, $\mathcal{L}_\xi$, of the bulk fields w.r.t. a bulk vector field ξ^μ is given by

$$\begin{aligned}\mathcal{L}_\xi g_{rr}&=\dot{\xi}^r\,,\\ \mathcal{L}_\xi g_{ri}&=\gamma_{ij}(\dot{\xi}^j+\partial^j\xi^r)\,,\\ \mathcal{L}_\xi g_{ij}&=L_\xi\gamma_{ij}+2K_{ij}\xi^r\sim L_\xi\gamma_{ij}+2\gamma_{ij}\xi^r\,,\end{aligned} \tag{3.26}$$

$$\begin{aligned}\mathcal{L}_\xi A_r&=A_j\dot{\xi}^j\,,\\ \mathcal{L}_\xi A_i&=L_\xi A_i+\xi^r\dot{A}_i\sim L_\xi A_i,\end{aligned} \tag{3.27}$$

$$\mathcal{L}_\xi\Phi^I=L_\xi\Phi^I+\xi^r\dot{\Phi}^I\sim L_\xi\Phi^I+(\Delta_I-d)\xi^r\Phi^I\,, \tag{3.28}$$

where L_ξ is the Lie derivative w.r.t. the transverse components ξ^i of the bulk vector field ξ. This bulk diffeomorphism, combined with a U(1) gauge transformation, preserves the gauge fixing (up to the desired order; see (3.39) below) provided $\mathcal{L}_\xi g_{rr}=\mathcal{L}_\xi g_{ri}=\mathcal{O}(e^{-dr})$ and $\mathcal{L}_\xi A_r+\dot{\alpha}=\mathcal{O}(e^{-(d+2)r})$. Integrating these conditions gives:

$$\begin{aligned}\xi^r&=\delta\sigma(x)+\mathcal{O}(e^{-dr})\,,\\ \xi^i&=\xi^i_o(x)+\partial_j\delta\sigma(x)\int_r^\infty dr'\gamma^{ji}(r',x)+\mathcal{O}(e^{-(d+2)r})\,,\end{aligned}$$

[8] We will often refer to the same quantity evaluated on Σ_{r_o} as the 'renormalized action', i.e. before the limit $r_o\to\infty$ is taken.

$$\alpha = \alpha_o(x) + \partial_i\delta\sigma(x)\int_r^\infty dr' A^i(r',x) + \mathcal{O}(e^{-(d+2)r})\,, \qquad (3.29)$$

where $\delta\sigma(x)$ and $\alpha_o(x)$ are arbitrary functions of the transverse coordinates and $\xi_o^i(x)$ is an arbitrary transverse vector field. For $\xi_o = 0$, this bulk diffeomorphism is precisely the 'Penrose-Brown-Henneaux (PBH) transformation' [26, 36] which induces a Weyl transformation on the conformal boundary [37, 18, 19]. Here, we will call a 'PBH transformation' the combined bulk diffeomorphism with $\xi_o = 0$ *and* the gauge transformation with $\alpha_o = 0$, which is required in order to preserve the gauge of the Maxwell field.

Next we determine which subset of (3.29) leaves invariant the renormalized action

$$I_{\rm ren} = \int_{\mathcal{M}_{r_o}} \mathbf{L} + \frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K + I_{\rm ct}\,, \qquad (3.30)$$

where

$$\mathbf{L} = \left(\frac{1}{2\kappa^2}R[g] + \mathcal{L}_{\rm m}\right) * \mathbf{1}\,, \qquad (3.31)$$

and $I_{\rm ct}$ is given by (3.24). Since $\mathbf{L}$ is covariant under diffeomorphisms and gauge invariant, we have

$$\delta_\xi\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = \mathrm{d}i_\xi\mathbf{L}\,, \qquad \delta_\alpha\mathbf{L} = 0\,, \qquad (3.32)$$

where we have used the identity $\mathcal{L}_\xi = i_\xi\mathrm{d} + \mathrm{d}i_\xi$ for the Lie derivative on forms. Hence,

$$\delta_{\xi,\alpha}I_{\rm ren} = \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\xi^r\left(\frac{1}{2\kappa^2}R[g] + \mathcal{L}_{\rm m}\right) + \frac{1}{\kappa^2}\delta_\xi\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K + \delta_{\xi,\alpha}I_{\rm ct}\,. \qquad (3.33)$$

Now, in the gauge we are using, the Ricci scalar of the bulk metric can be expressed as

$$R[g] = R + K^2 - K_{ij}K^{ij} - \frac{2}{\sqrt{-\gamma}}\partial_r(\sqrt{-\gamma}K)\,, \qquad (3.34)$$

Moreover, for the diffeomorphisms given by (3.29) a short computation gives

$$\delta_\xi\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K = \int_{\Sigma_{r_o}} d^dx\,\xi^r\partial_r(\sqrt{-\gamma}K)\,, \qquad (3.35)$$

and hence

$$\delta_{\xi,\alpha}I_{\rm ren} = \frac{1}{2\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\xi^r\left(R + K^2 - K_{ij}K^{ij} + 2\kappa^2\mathcal{L}_{\rm m}\right) + \delta_{\xi,\alpha}I_{\rm ct}\,. \qquad (3.36)$$

The last term takes the form

$$\delta_{\xi,\alpha}I_{\rm ct} = \int_{\Sigma_{r_o}} d^dx\left(\hat{\pi}^{ij}_{\rm ct}\delta_\xi\gamma_{ij} + \hat{\pi}_{\rm ct\,I}\delta_\xi\Phi^I + \hat{\pi}^i_{\rm ct}(\delta_\xi A_i + \partial_i\alpha)\right), \qquad (3.37)$$

where we put hats on the counterterm momenta to emphasize that they should be viewed as predetermined local functionals of the induced fields as opposed to the asymptotic behavior of the radial derivative of the on-shell induced fields. Inserting now the transformation (3.29) and using the second equation in (A.2) and the first equation in (A.4), which the counterterms satisfy by construction, we are left with

$$\delta_{\xi,\alpha}I_{\rm ren} = \int_{\Sigma_{r_o}} d^dx\,\xi^r\Big\{\frac{1}{2\kappa^2}\sqrt{-\gamma}\left(R + K^2 - K_{ij}K^{ij} + 2\kappa^2\mathcal{L}_{\rm m}\right) +$$

$$+\left(\hat{\pi}_{\rm ct}^{ij}2K_{ij}+\hat{\pi}_{\rm ct\,I}\dot{\Phi}^I+\hat{\pi}_{\rm ct}^i\dot{A}_i\right)\Bigg\}\,. \tag{3.38}$$

Using the form of the boundary conditions (3.25) one finds that the leading order divergent term cancels and the terms inside the curly brackets are of order $e^{(d-1)r}$. We therefore conclude that a transformation (3.29) that leaves the renormalized action invariant must have

$$\xi^r=\mathcal{O}(e^{-dr})\,,\qquad \dot{\xi}^i=-\partial^i\xi^r+\mathcal{O}(e^{-(d+2)r})=\mathcal{O}(e^{-(d+2)r})\,. \tag{3.39}$$

This leaves us with $\xi^i=\xi^i_o(x)$ and $\alpha=\alpha_o(x)$, up to sufficiently high order in e^{-r} as $r\to\infty$.

In fact, as is well known, the PBH transformation, i.e. the part of the transformation (3.29) that is driven by $\delta\sigma(x)$, induces a Weyl transformation on the boundary and even the *on-shell* renormalized action is not invariant under such transformations unless the anomaly vanishes. To see this let us first rewrite the Hamilton constraint (first equation in (A.2)) as

$$\frac{1}{2\kappa^2}\sqrt{-\gamma}\left(R+K^2-K_{ij}K^{ij}+2\kappa^2\mathcal{L}_{\rm m}\right)=\pi^{ij}2K_{ij}+\pi_I\dot{\Phi}^I+\pi^i\dot{A}_i\,. \tag{3.40}$$

Then, using the trace Ward identity (3.22), (3.38) becomes on-shell

$$\delta_{\xi,\alpha}I_{\rm ren}^{\rm on-shell}=\int_{\Sigma_{r_o}}d^dx\sqrt{-\gamma}\xi^r\mathcal{A}\,. \tag{3.41}$$

3.4 Variational problem

We investigate in this section under which conditions the variational problem is well-posed, i.e. under which conditions the boundary terms in the variation of the action cancel so that $\delta I=0$ (under generic variation) implies the field equations and vice versa.

Let n^μ be the outward unit normal to the hypersurfaces Σ_r. Using (3.7) and the definition of the radial momenta (3.9) one easily finds that the pullback of $\mathbf{\Theta}$ onto Σ_r is given by[9]

$$\begin{aligned}\mathbf{\Theta}&=-n_\mu v^\mu *_\Sigma \mathbf{1}\\ &=\left(-\frac{1}{\kappa^2}\delta(\sqrt{-\gamma}K)+\pi^{ij}\delta\gamma_{ij}+\pi^i\delta A_i+\pi_I\delta\Phi^I\right)d\mu\,,\end{aligned} \tag{3.42}$$

where $\sqrt{-\gamma}d\mu\equiv *_\Sigma\mathbf{1}$, and $*_\Sigma$ denotes the Hodge dual w.r.t. Σ_r. We thus arrive at the well-known fact [38] that the Gibbons-Hawking term is sufficient to render the variational problem well-defined when all induced fields at the boundary are kept fixed, i.e.

$$\delta\gamma_{ij}=0\,,\qquad \delta A_i=0\,,\qquad \delta\Phi^I=0\qquad \text{on}\qquad \Sigma_{r_o}\,. \tag{3.43}$$

These boundary conditions are perfectly acceptable for the regulated manifold with boundary Σ_{r_o} at finite r_o, since the bulk fields uniquely induce fields on Σ_{r_o}. However, as $\Sigma_{r_o}\to\partial\mathcal{M}$ this is no longer the case. The induced fields generically diverge (or vanish)

[9] Up to an exact term d($*\mathbf{y}$), where $y^\mu=\frac{1}{2\kappa^2}n^\rho g^{\mu\sigma}\delta g_{\rho\sigma}$ vanishes for variations that preserve the gauge fixing.

in this limit and the bulk fields only determine the *conformal class* of the boundary fields. It is therefore not possible to impose the above boundary conditions on the conformal boundary. At most, one can demand that the boundary fields are kept fixed up to a Weyl transformation, namely

$$\delta\gamma_{ij} = 2\gamma_{ij}\delta\sigma\,, \qquad \delta A_i = 0\,, \qquad \delta\Phi^I = (\Delta_I - d)\Phi^I\delta\sigma \qquad \text{on} \qquad \partial\mathcal{M}\,. \tag{3.44}$$

To implement these weaker boundary conditions we insert the expansions (3.17) into (3.42) and use (3.14) to get

$$\begin{aligned}\mathbf{\Theta} &= \Big\{ -\frac{1}{\kappa^2}\delta(\sqrt{-\gamma}K) - (\pi_{\rm ct}^{ij}\delta\gamma_{ij} + \pi_{\rm ct}^{i}\delta A_i + \pi_{{\rm ct}I}\delta\Phi^I) + \\ &\qquad + \sqrt{-\gamma}({\pi_{(d)}}^{ij}\delta\gamma_{ij} + {\pi_{(d)}}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\Phi^I) + \cdots \Big\} d\mu \\ &= \Big\{ \delta\left(-\frac{1}{\kappa^2}\sqrt{-\gamma}[K - (K-\lambda)_{\rm ct}]\right) + \\ &\qquad + \sqrt{-\gamma}({\pi_{(d)}}^{ij}\delta\gamma_{ij} + {\pi_{(d)}}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\Phi^I) + \cdots \Big\} d\mu\,. \end{aligned} \tag{3.45}$$

Hence,

$$\begin{aligned}\int_{\Sigma_{r_o}} \mathbf{\Theta} &= \delta\left(-\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K - I_{\rm ct}\right) + \\ &\qquad + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\left[{\pi_{(d)}}^{ij}\delta\gamma_{ij} + {\pi_{(d)}}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\Phi^I + \cdots\right], \end{aligned} \tag{3.46}$$

where the dots denote terms of higher dilatation weight which do not survive after the regulator is removed and $I_{\rm ct}$ is *local* in the boundary fields. Finally we insert the boundary conditions (3.44) and use the diffeomorphism and trace Ward identities (3.21) and (3.22) to arrive at

$$\int_{\Sigma_{r_o}} \mathbf{\Theta} = \delta\left(-\frac{1}{\kappa^2}\int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}K - I_{\rm ct}\right) + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\mathcal{A}\delta\sigma\,. \tag{3.47}$$

It follows that

$$\delta I_{\rm ren}^{\rm on-shell} = \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\mathcal{A}\delta\sigma\,. \tag{3.48}$$

Notice that $\mathcal{A}$ is uniquely determined from boundary data. Furthermore, its integral is conformally invariant. It follows that $\mathcal{A}$ is a conformal density of weight d modulo total derivatives.

There are three cases to discuss now.

1. The unintegrated anomaly vanishes identically:

$$\mathcal{A} \equiv 0\,. \tag{3.49}$$

   This is the case, for instance, for pure AAdS gravity in even dimensions. Our analysis shows that the variational problem in this case is well-posed, provided we augment the Gibbons-Hawking term by the usual counterterms.

2. The integrated anomaly vanishes for a particular conformal class $[\mathtt{g}_{(0)}]$,

$$A[\mathtt{g}_{(0)}] \equiv \int_{\partial\mathcal{M}} d^d x \sqrt{-\mathtt{g}_{(0)}} \mathcal{A}[\mathtt{g}_{(0)}] = 0\,. \tag{3.50}$$

   This is the case, for instance, for pure AAdS gravity in odd dimensions with the conformal class represented by the standard metric on the boundary $\mathbb{R} \times \mathbf{S}^{d-1}$. When (3.50) holds the anomaly density does not necessarily vanish and so the variational problem with the boundary conditions (3.44) is not well-defined in general since the variation of the action generically contains a non-vanishing boundary term. Nevertheless, the vanishing of the integrated anomaly guarantees that there exists a representative $\mathtt{g}_{(0)}$ of the conformal class $[\mathtt{g}_{(0)}]$ for which the anomaly density, $\mathcal{A}$, is zero. For instance, for pure AAdS gravity in odd dimensions such a representative is the standard metric on $\mathbb{R} \times \mathbf{S}^{d-1}$. Hence one can pick a suitable defining function which induces this particular representative. In practice this means that we want to perform a PBH transformation such that the resulting radial coordinate acts as a defining function which induces the desired representative. We then consider the variational problem *around this gauge* that corresponds to the privileged representative of the conformal structure at the boundary for which the anomaly density vanishes. However, this ensures only that the first order variation of the action will contain no boundary terms. To make the variational problem well-defined to all orders one is forced to break the bulk diffeomorphisms which induce a Weyl transformation on the boundary and consider variations of the bulk fields which preserve a particular representative of the conformal class. In other words, in this case, in order to make the variational problem well-defined to all orders we *must* impose the boundary conditions,

$$\delta\mathtt{g}_{(0)ij} = 0\,, \qquad \delta\mathtt{A}_{(0)i} = 0\,, \qquad \delta\phi_{(0)} = 0\,, \tag{3.51}$$

   where $\mathtt{g}_{(0)ij}(x)$ is the chosen representative of the conformal structure and $\mathtt{A}_{(0)i}(x)$ and $\phi_{(0)}(x)$ are the leading terms in the asymptotic expansion of the bulk gauge and scalar fields, respectively,

$$A_i(x,r) = \mathtt{A}_{(0)i}(x)(1+\mathcal{O}(e^{-r}))\,, \qquad \Phi(x,r) = \phi_{(0)}(x)e^{-(d-\Delta_I)r}(1+\mathcal{O}(e^{-r}))\,. \tag{3.52}$$

   As we have seen, however, this is only possible if one breaks certain bulk diffeomorphisms.

3. The integrated anomaly is non-zero. In this case, to ensure that the variational problem is well defined already at leading order, we have to pick a representative and allow only variations that preserve the corresponding gauge.

To summarize, we have seen that bulk covariance in AlAdS spaces requires that we formulate the variational problem with the boundary conditions (3.44) instead of the stronger (3.43). The counterterms are *essential* in making the variational problem well-defined with such boundary conditions and are exactly on the same footing with the

Gibbons-Hawking term. However, when the unintegrated anomaly does not vanish identically, the variational problem can only be well-defined (to all orders) with the boundary conditions (3.51), which can only be imposed if certain bulk diffeomorphisms are broken. The counterterms in this case guarantee that the on-shell action has a well-defined transformation under the broken diffeomorphisms. The transformation is given precisely by the anomaly.

4. Holographic charges are Noether charges

4.1 Conserved charges associated with asymptotic symmetries

We have seen in section 3.3 that the renormalized action is invariant under bulk diffeomorphisms and U(1) gauge transformations that asymptotically take the form (3.29) provided $\xi^r = \mathcal{O}(e^{-dr})$. Moreover, requiring that such transformations preserve the boundary conditions (3.44) constrains ξ^i to be an asymptotic conformal Killing vector, i.e. to asymptotically approach a boundary conformal Killing vector (see appendix B for the precise definition). When the anomaly does not vanish, however, we impose the boundary conditions (3.51) which are only preserved if ξ^i is a boundary *Killing* vector (as opposed to asymptotic conformal Killing vector).

We now apply Noether's theorem to extract the conserved currents and charges associated with these asymptotic symmetries. To this end we first consider the following field variations:

$$\delta_1\psi = f_1(r,x)\mathcal{L}_\xi\psi\,, \qquad \delta_2\psi = f_2(r,x)\delta_\alpha\psi\,, \tag{4.1}$$

where $f_1(r,x)$, $f_2(r,x)$ are arbitrary functions on $\mathcal{M}$ which reduce to functions $\bar{f}_1(x)$ and $\bar{f}_2(x)$ respectively on $\partial\mathcal{M}$, ξ^i is an asymptotic conformal Killing vector of the induced fields on Σ_r and α is a gauge parameter which asymptotically tends to a constant. These, transformations are not a symmetry of the renormalized action unless f_1 and f_2 are constants, but they preserve the boundary conditions (3.44) for arbitrary f_1, f_2. Varying the renormalized action, whose general variation is given by

$$\delta I_{\rm ren} = \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta\psi + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}\left(\pi_{(d)}{}^{ij}\delta\gamma_{ij} + \pi_{(d)}{}^{i}\delta A_i + \pi_{(\Delta_I)I}\delta\Phi^I\right), \tag{4.2}$$

with respect to such field variations we will now derive the conserved Noether charges.

Electric charge. Let us first consider the transformation $\delta_2\psi$ and derive the corresponding conserved current. Since $\delta_\alpha\mathbf{L} = 0$, we have from (3.2)

$$\mathbf{E}\delta_\alpha\psi = -\mathrm{d}\mathbf{\Theta}(\psi,\delta_\alpha\psi)\,. \tag{4.3}$$

Hence,

$$\delta_2 I_{\rm ren} = -\int_{\mathcal{M}_{r_o}} f_2\mathrm{d}\mathbf{\Theta}(\psi,\delta_\alpha\psi) + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma}f_2\pi_{(d)}{}^{i}\partial_i\alpha\,, \tag{4.4}$$

But α is asymptotically constant and so the boundary term vanishes. Hence, on-shell the bulk integral on the r.h.s. must vanish for arbitrary f_2, which leads to the conservation law for the U(1) current

$$\mathbf{J}_\alpha \equiv \mathbf{\Theta}(\psi,\delta_\alpha\psi)\,. \tag{4.5}$$

Since on-shell $\mathbf{J}_\alpha$ is closed, it is locally exact. In fact one easily finds

$$\mathbf{J}_\alpha = \mathrm{d}\mathbf{Q}_\alpha \,, \tag{4.6}$$

where $\mathbf{Q}_\alpha = -\alpha * \mathcal{F}$ and $\mathcal{F}_{\mu\nu} = U(\Phi)F_{\mu\nu}$. Then, given a Cauchy surface C, the conserved Noether charge is given by[10]

$$Q = \int_C \mathbf{J}_\alpha = -\int_{\partial\mathcal{M}\cap C} *\mathcal{F} \,, \tag{4.7}$$

where we have assumed without loss of generality that $\alpha \to 1$ on $\partial\mathcal{M}$. One can check that this charge is conserved, i.e. independent of the Cauchy surface C, which follows immediately from the field equation

$$\mathrm{d} * \mathcal{F} = 0 \,. \tag{4.8}$$

Charges associated with boundary conformal isometries. The same argument can be applied to derive the conserved currents and Noether charges associated with asymptotic conformal isometries of the induced fields. Again from (3.2) we have

$$\mathbf{E}\mathcal{L}_\xi\psi = \mathrm{d}\left(i_\xi\mathbf{L} - \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi)\right) . \tag{4.9}$$

Hence, defining the current

$$\mathbf{J}[\xi] \equiv \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\mathbf{L} \,, \tag{4.10}$$

we get

$$\delta_1 I_{\mathrm{ren}} = -\int_{\mathcal{M}_{r_o}} f_1 \mathrm{d}\mathbf{J}[\xi] + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma} f_1 \left(\pi_{(d)}{}^{ij} L_\xi\gamma_{ij} + \pi_{(d)} L_\xi A_i + \pi_{(\Delta_I)I} L_\xi \Phi^I\right) . \tag{4.11}$$

Since ξ^i is an asymptotically conformal Killing vector, it follows that

$$\delta_1 I_{\mathrm{ren}} = -\int_{\mathcal{M}_{r_o}} f_1 \mathrm{d}\mathbf{J}[\xi] + \int_{\Sigma_{r_o}} d^dx\sqrt{-\gamma} f_1 \left(2\pi_{(d)}{}_i^i + (\Delta_I - d)\pi_{(\Delta_I)I}\Phi^I\right) \frac{1}{d} D_i\xi^i \,. \tag{4.12}$$

Now, evaluating the l.h.s. using (3.48) and the r.h.s. using the trace Ward identity (3.22), we deduce that on-shell the bulk integral vanishes, which leads to the conservation law

$$\mathrm{d}\mathbf{J}[\xi] = 0 \,. \tag{4.13}$$

Hence, $\mathbf{J}[\xi]$ is locally exact, $\mathbf{J}[\xi] = \mathrm{d}\mathbf{Q}[\xi]$, and it is easily shown that

$$\mathbf{Q}[\xi] = -\frac{1}{\kappa^2} * \mathbf{\Xi}[\xi] \,, \tag{4.14}$$

where the 2-form $\mathbf{\Xi}$ is given by

$$\Xi_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]} + \kappa^2 U(\Phi) F_{\mu\nu} A_\rho \xi^\rho \,. \tag{4.15}$$

[10]Throughout this article we use the convention about the (relative) orientation $\epsilon_{rti_2\ldots i_d} \equiv \epsilon_{ti_2\ldots i_d} = +1$. The minus sign in the definition of the electric charge is included to compensate for this choice of orientation, which is opposite from the conventional one.

However, $\mathbf{J}[\xi]$ is not the full Noether current in this case as there is an extra contribution with support on Σ_{r_o}. To derive the correct form of the current we use (3.45) to rewrite (4.11) as

$$\begin{aligned}
\delta_1 I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1 \mathbf{J}[\xi] + \\
&\quad + \int_{\Sigma_{r_o}} d^d x \sqrt{-\gamma} f_1 \left({\pi_{(d)}}^{ij} L_\xi \gamma_{ij} + \pi_{(d)} L_\xi A_i + \pi_{(\Delta_I)I} L_\xi \Phi^I \right) \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] + \int_{\Sigma_{r_o}} f_1 i_\xi \mathbf{L} + \int_{\Sigma_{r_o}} d^d x f_1 \delta_\xi \left(\frac{1}{\kappa^2} \sqrt{-\gamma} [K - (K-\lambda)_{\rm ct}] \right)
\end{aligned} \tag{4.16}$$

Since ξ is tangent to Σ_{r_o}, the second term vanishes. To put the last term in the desired form, we define the d-form

$$\mathbf{B} \equiv -\frac{1}{\kappa^2} [K - (K-\lambda)_{\rm ct}] *_\Sigma \mathbf{1} \tag{4.17}$$

on Σ_r which is covariant w.r.t. diffeomorphisms within Σ_r. Using the identity $L_\xi = \bar{\mathrm{d}} i_\xi + i_\xi \bar{\mathrm{d}}$ on forms, $\bar{d}$ being the exterior derivative on Σ_r, we obtain

$$\begin{aligned}
\delta_1 I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1 \delta_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] - \int_{\Sigma_{r_o}} f_1 \bar{\mathrm{d}} i_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] + \int_{\Sigma_{r_o}} \bar{\mathrm{d}} f_1 \wedge i_\xi \mathbf{B} \\
&= \int_{\mathcal{M}_{r_o}} \mathrm{d}f_1 \wedge \mathbf{J}[\xi] + \int_{\mathcal{M}_{r_o}} \rho(\Sigma_{r_o}) \wedge \mathrm{d}f_1 \wedge i_\xi \mathbf{B} \,,
\end{aligned} \tag{4.18}$$

where $\rho(\Sigma_r)$ is a one-form with delta function support on Σ_r, known as the Poincaré dual of Σ_r in $\mathcal{M}$. Therefore, the full Noether current is

$$\tilde{\mathbf{J}}[\xi] \equiv \mathbf{J}[\xi] - \rho(\Sigma_{r_o}) \wedge i_\xi \mathbf{B} \,. \tag{4.19}$$

Given a Cauchy surface C, we now define the Noether charge

$$Q[\xi] \equiv \int_C \tilde{\mathbf{J}}[\xi] = \int_{\partial\mathcal{M} \cap C} (\mathbf{Q}[\xi] - i_\xi \mathbf{B}) \,. \tag{4.20}$$

If C and C' are two Cauchy surfaces whose intersection with $\partial\mathcal{M}$ bounds a domain $\Delta \subset \partial\mathcal{M}$, then Stokes' theorem and the conservation law (4.13) imply

$$\begin{aligned}
Q_C[\xi] - Q_{C'}[\xi] &= \int_{\Delta \subset \partial\mathcal{M}} (\mathbf{J}[\xi] - \mathrm{d} i_\xi \mathbf{B}) \\
&= \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \left({\pi_{(d)}}^{ij} L_\xi \gamma_{ij} + \pi_{(d)} L_\xi A_i + \pi_{(\Delta_I)I} L_\xi \Phi^I \right) \\
&= \int_{\partial\mathcal{M}} d^d x \sqrt{-\gamma} \left(\pi_{(d)}{}^i_i + (\Delta_I - d) \pi_{(\Delta_I)I} \Phi^I \right) \frac{1}{d} D_i \xi^i
\end{aligned}$$

$$= \int_{\partial\mathcal{M}} d^d x\sqrt{-\gamma}\mathcal{A}\frac{1}{d}D_i\xi^i\,. \tag{4.21}$$

Therefore, if the anomaly vanishes, this charge is conserved for any asymptotic conformal Killing vector. However, if the anomaly is non-zero, it is only conserved for symmetries associated with boundary Killing vectors.

4.2 Holographic charges

Let us now derive an alternative form of the conserved charges by considering instead of (4.1) the following variations:

$$\delta_1'\psi = \mathcal{L}_{\epsilon\xi}\psi\,, \qquad \delta_2'\psi = \delta_\alpha\psi\,, \tag{4.22}$$

where ξ^i is again an asymptotic conformal Killing vector but now α and ϵ reduce to arbitrary functions on Σ_{r_o}. In contrast to (4.1), these field variations *are* a symmetry of the action, but they violate the boundary conditions (3.44).

Since these are symmetries of the renormalized action we have

$$\begin{aligned} 0 = \delta_2' I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_2'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\pi_{(d)}{}^i\partial_i\alpha \\ &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_2'\psi - \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\alpha\partial_i\pi_{(d)}{}^i\,. \end{aligned} \tag{4.23}$$

But now α is arbitrary and so we conclude that on-shell we must have

$$\partial_i\pi_{(d)}{}^i = 0\,, \tag{4.24}$$

which also follows immediately from the first equation in (A.4). Hence the quantity

$$\mathcal{Q} \equiv -\int_{\partial\mathcal{M}\cap C} d\sigma_i\pi_{(d)}{}^i\,, \tag{4.25}$$

defines a conserved charge, namely the *holographic* electric charge.

Similarly,

$$\begin{aligned} 0 = \delta_1' I_{\rm ren} &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_1'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\left(\pi_{(d)}{}^{ij}L_{\epsilon\xi}\gamma_{ij} + \pi_{(d)}{}^i L_{\epsilon\xi}A_i + \pi_{(\Delta_I)I}L_{\epsilon\xi}\Phi^I\right) \\ &= \int_{\mathcal{M}_{r_o}} \mathbf{E}\delta_1'\psi + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\epsilon\left(\pi_{(d)}{}^{ij}L_{\xi}\gamma_{ij} + \pi_{(d)}{}^i L_{\xi}A_i + \pi_{(\Delta_I)I}L_{\xi}\Phi^I\right) + \\ &\quad + \int_{\Sigma_{r_o}} d^d x\sqrt{-\gamma}\left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j D_i\epsilon\,. \end{aligned} \tag{4.26}$$

Therefore, after integration by parts in the last term and using the fact that ϵ is arbitrary, we conclude that on-shell we must have

$$\begin{aligned} D_i\left[(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j)\xi^j\right] &= \pi_{(d)}{}^{ij}L_\xi\gamma_{ij} + \pi_{(d)}{}^i L_\xi A_i + \pi_{(\Delta_I)I}L_\xi\Phi^I \\ &= \left(2\pi_{(d)}{}^i_i + (\Delta_I - d)\pi_{(\Delta_I)I}\Phi^I\right)\frac{1}{d}D_i\xi^i \end{aligned}$$

$$= \mathcal{A}\frac{1}{d}D_i\xi^i\,, \tag{4.27}$$

where we have used the trace Ward identity (3.22) in the last step. Hence the quantity

$$\mathcal{Q}[\xi] \equiv \int_{\partial\mathcal{M}\cap C} d\sigma_i \left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j\,, \tag{4.28}$$

defines a holographic conserved charge associated with every asymptotic conformal Killing vector, if the anomaly vanishes, or every boundary Killing vector, if the anomaly does not vanish.

From the above analysis we have obtained two apparently different expressions for the conserved charges associated with every asymptotic symmetry. However, we show in the the following lemma that the two expressions for the conserved charges are equivalent.

Lemma 1. *Let ψ denote an AlAdS solution of the bulk equations of motion possessing an asymptotic timelike Killing vector k and possibly a set of $N-1$ asymptotic spacelike Killing vectors m_α with closed orbits, forming a maximal set of commuting asymptotic isometries. In adapted coordinates such that $k=\partial_t$ and $m_\alpha = \partial_{\phi^\alpha}$ the background ψ is independent of the coordinates $x^a=\{t,\phi^\alpha\}$. Then,*

(i)

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \pi_{(d)}{}^i = \int_{\partial\mathcal{M}\cap C} *\mathcal{F}\,. \tag{4.29}$$

(ii) If in addition the background metric and gauge field take asymptotically the form

$$d\bar{s}^2 \equiv \gamma_{ij}dx^i dx^j = \tau_{ab}dx^a dx^b + \sigma_{ij}d\tilde{x}^i d\tilde{x}^j\,, \qquad \mathbf{A} \equiv A_i dx^i = A_a dx^a\,, \tag{4.30}$$

where τ_{ab}, σ_{ij} and A_a depend only on the rest of the transverse coordinates $\tilde{x}^i$ as well as the radial coordinate r, then, for any asymptotic conformal Killing vector ξ,

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j = -\int_{\partial\mathcal{M}\cap C} \left(\mathbf{Q}[\xi] - i_\xi \mathbf{B}\right). \tag{4.31}$$

A proof of this lemma can be found in appendix C. However, a few comments are in order regarding the condition (4.30) we have assumed in order to prove the second part of the lemma. Firstly, as can be seen from the explicit proof, it is only required in order to show equivalence of the charges for true asymptotic *conformal* isometries, i.e. with *non-zero* conformal factor. Otherwise, this condition is not used in the proof. Secondly, in certain special cases the fact that the background takes the form (4.30) turns out to be a consequence of the existence of the set of commuting isometries and the field equations.

More specifically, the condition that the background takes the form (4.30) is closely related to the integrability of the $D-N$-dimensional submanifolds orthogonal to k and m_α. In particular, it was shown in [39], Theorem 7.7.1, using Frobenius' theorem, that for pure gravity in four dimensions, the 2-planes orthogonal to a timelike isometry k and a rotation m are integrable, and hence the metric takes the form (4.30). This result can be easily extended to include a Maxwell field as well as scalar fields in four dimensions [40, 41]. More recently, this result was generalized for pure gravity in D dimensions and $D-2$ orthogonal (non-orthogonal) commuting isometries in [42] ([43]). It appears that these results cannot be generalized in a straightforward way to include gauge fields for $D > 4$, or for less than $D-2$ commuting isometries in D dimensions. Obviously the restriction to $D-2$ commuting isometries is too strong for our purposes since even AdS_D only has $[(D+1)/2]$ commuting isometries, which is less than $D-2$ for $D > 5$.

Despite the fact that we lack a general proof of (4.30) as a consequence of the presence of the commuting isometries and the field equations, this condition is satisfied by a very wide range of AlAdS spacetimes, including Taub-Nut-AdS and Taub-Bolt-AdS [44, 45]. It would be very interesting to determine what are the most general conditions so that (4.30) holds.

4.3 Wald hamiltonians

We now give a third derivation of the conserved charges as 'hamiltonians' on the covariant phase space [47, 48, 23, 24]. Some results relevant to this section are collected in appendix D.

Let ξ be an asymptotic conformal Killing vector and α an asymptotically constant gauge transformation, namely

$$\mathcal{L}_\xi \psi = \mathcal{L}_{\hat{\xi}} \psi + \delta_{\hat{\alpha}} \psi + \mathcal{O}(e^{-s_+ r}) , \qquad \delta_\alpha \psi = \mathcal{O}(e^{-s^+ r}) , \tag{4.32}$$

where $\hat{\alpha}$, $\hat{\xi}$ and s^+ are given in appendix B. The 'hamiltonians' which generate these symmetries in phase space must satisfy Hamilton's equations, which in the covariant phase space formalism take the form

$$\delta H_\xi = \Omega_C(\psi, \delta\psi, \mathcal{L}_\xi \psi) , \qquad \delta H_\alpha = \Omega_C(\psi, \delta\psi, \delta_\alpha \psi) , \tag{4.33}$$

where the pre-symplectic form Ω_C is defined in (D.15). The hamiltonians exist if these equations can be integrated in configuration space to give H_ξ and H_α. As is discussed in appendix D, the symplectic form is independent of the Cauchy surface used to define it if the anomaly vanishes or if the variations are associated with boundary Killing vectors. It follows that the corresponding hamiltonians are conserved, provided the 'integration' constant is also independent of the Cauchy surface. We further discuss this issue below.

Let us first consider H_α which can be obtained very easily. Using the result for the symplectic form in (D.9) we have

$$\delta H_\alpha = \int_{\partial\mathcal{M}\cap C} \delta \mathbf{Q}_\alpha , \tag{4.34}$$

and hence, up to a constant,

$$H_\alpha = \int_{\partial\mathcal{M}\cap C} \mathbf{Q}_\alpha = - \int_{\partial\mathcal{M}\cap C} *\mathcal{F} , \tag{4.35}$$

taking $\alpha \to 1$ asymptotically. Therefore, once again, we have derived the conserved electric charge.

Consider next H_ξ. From (D.8) we have

$$\delta H_\xi = \int_{\partial\mathcal{M}\cap C} (\delta \mathbf{Q}[\xi] - i_\xi \mathbf{\Theta})\,. \tag{4.36}$$

This equation has a non-trivial integrability condition. Applying a second variation and using the commutativity of two variations, $\delta_1\delta_2 - \delta_2\delta_1 = 0$, we obtain the integrability condition [24]

$$\int_{\partial\mathcal{M}\cap C} i_\xi \omega(\psi, \delta\psi_1, \delta\psi_2) = 0\,. \tag{4.37}$$

Since ξ is tangent to Σ_r, from (D.18) follows that this is equivalent to

$$\int_{\partial\mathcal{M}\cap C} d^{d-1}x\xi^t \left\{\delta_2(\sqrt{-\gamma}\mathcal{A})\delta_1\sigma - 1 \leftrightarrow 2\right\} = 0\,. \tag{4.38}$$

Therefore, if the anomaly vanishes, a hamiltonian associated to any asymptotic CKV ξ exists. However, if there is an anomaly and $\xi^t \neq 0$, then a hamiltonian for ξ exists only if the stronger boundary condition (3.51) is used - i.e. a particular representative of the conformal class is kept fixed - in agreement with the analysis of the variational problem.

The same conclusion can be drawn by trying to integrate (4.36) directly. This is possible provided we can find a d-form $\mathbf{B}$ such that

$$\int_{\partial\mathcal{M}\cap C} i_\xi \mathbf{\Theta} = \delta \int_{\partial\mathcal{M}\cap C} i_\xi \mathbf{B}\,. \tag{4.39}$$

Once such a form is found, then H_ξ exists and it is given by

$$H_\xi = \int_{\partial\mathcal{M}\cap C} (\mathbf{Q}[\xi] - i_\xi \mathbf{B})\,. \tag{4.40}$$

However, since ξ is tangent to Σ_r, we can use (3.45), the boundary conditions (3.44) and the trace Ward identity (3.22) to obtain

$$\int_{\Sigma_{r_o}\cap C} i_\xi \mathbf{\Theta} = \frac{1}{\kappa^2}\delta \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i [K - (K-\lambda)_{\rm ct}] - \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \mathcal{A}\delta\sigma\,. \tag{4.41}$$

Therefore, if $\xi^t \neq 0$, then such a form exists for the boundary conditions (3.44) provided the anomaly vanishes, in complete agreement with the conclusion from the integrability condition. Moreover, (4.41) shows that when such a $\mathbf{B}$ exists it coincides with $\mathbf{B}$ in (4.17) and hence, the corresponding hamiltonian is precisely the Noether charge (4.20).

Notice that the Wald hamiltonians are only defined up to quantities in the kernel of the variations. In particular, when integrating (4.39) to obtain (4.40), one can add to H_ξ an integral of a local density constructed only from boundary data and the asymptotic conformal Killing vector ξ. This 'integration constant' is constrained by the fact that the hamiltonians should be conserved. In particular, if H_ξ is a Wald hamiltonian, so is

$$H'_\xi = H_\xi + \int_{\partial\mathcal{M}\cap C} d\sigma_i H^i_j \xi^j\,, \tag{4.42}$$

provided H^i_j is constructed locally from boundary data, has dilatation weight d, and it is covariantly conserved.

In fact such ambiguity is present in $AAdS_{2k+1}$ spacetimes and has caused some confusion in the literature. $AAdS_{2k+1}$ spacetimes are special in that the boundary is conformally flat, and even-dimensional conformally flat spacetimes admit local covariantly conserved stress energy tensors. This is true in all even dimensions, as we discuss in appendix E. The best known case is the four dimensional one: the tensor

$$H^i_j \equiv \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3} R R^i_j + \frac{1}{2} R^k_l R^l_k \delta^i_j - \frac{1}{4} R^2 \delta^i_j\right) \tag{4.43}$$

is covariantly conserved provided the metric is conformally flat. This tensor is well-known from studies of quantum field theories in curved backgrounds, see [49] (where it is called ${}^3H_{\mu\nu}$). It has been called 'accidentally conserved' in [49] because it is not the limit of a local tensor that is conserved in non-conformally flat spacetimes and cannot be derived by varying a local term. The same tensor is the holographic stress energy tensor of[11] AdS_5 [18]!

This tensor also appeared recently in comparisons between the 'conformal mass' of [50] and the holographic mass, see [50, 9] and appendix E. It follows that the conserved charges according to both definitions are Wald hamiltonians. It also follows from this discussion that the conformal mass is the mass of the spacetime *relative* to the AdS background. Furthermore, we conclude that the definition of [1] does not extend to general AlAdS spacetimes since H_{ij} is not covariantly conserved when the boundary metric is not conformally flat and we already know that the holographic charges are conserved for general AlAdS (and as shown in this section are also Wald hamiltonians).

5. The first law of black hole mechanics

The above detailed description of the conserved charges allows us to study the thermodynamics of AlAdS black hole spacetimes quite generically. In particular, we will consider a black hole solution of (3.1) possessing a timelike Killing vector k and possibly a set of spacelike isometries with closed orbit forming a maximal set of commuting isometries as in lemma 1, but here we require that these isometries be exact and not just asymptotic. The form (4.30) of the metric then implies that these bulk isometries correspond to boundary isometries and not merely asymptotic boundary conformal isometries. Moreover, we will assume that the event horizon, $\mathcal{N}$, of the black hole is a non-degenerate bifurcate Killing horizon of a timelike (outside the horizon) Killing vector χ such that the surface gravity, $\hat{\kappa}$, of the horizon is given by

$$\hat{\kappa}^2 = -\frac{1}{2}\nabla^\mu \chi^\nu \nabla_\mu \chi_\nu |_{\mathcal{N}} \,. \tag{5.1}$$

The inverse temperature, β, then is

$$\beta = T^{-1} = \frac{2\pi}{\hat{\kappa}} \,. \tag{5.2}$$

Let us begin with a lemma which is central to our analysis.

[11]More precisely, (4.43) is the holographic stress energy tensor associated to a bulk solution that is conformally flat, see (3.20) of [18]; all such solutions are locally isometric to AdS_5.

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Lemma 2. *Let ξ be a bulk Killing vector, I the renormalized on-shell euclidean action and $\mathcal{H} = \mathcal{N} \cap C$ the intersection of the horizon with the Cauchy surface. Let also t be the adapted coordinate to the timelike isometry k so that $k = \partial_t$.*

(i) If $\xi^t = 1$, then[12]

$$\beta \mathcal{Q}[\xi] - I = -\beta \int_{\mathcal{H}} \mathbf{Q}[\xi] \,. \tag{5.3}$$

(ii) If $\xi^t = 0$, then

$$\mathcal{Q}[\xi] = -\int_{\mathcal{H}} \mathbf{Q}[\xi] \,. \tag{5.4}$$

Proof. By Stokes' theorem[13]

$$\begin{aligned} \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\xi] &= \int_C \mathrm{d}\mathbf{Q}[\xi] + \int_{\mathcal{H}} \mathbf{Q}[\xi] \\ &= \int_C \left(\mathbf{\Theta}(\psi, \mathcal{L}_\xi \psi) - i_\xi \mathbf{L}\right) + \int_{\mathcal{H}} \mathbf{Q}[\xi] \\ &= -\int_C i_\xi \mathbf{L} + \int_{\mathcal{H}} \mathbf{Q}[\xi] \,. \end{aligned} \tag{5.5}$$

Now, (3.10) and the fact that the background is stationary allow us to write

$$\int_C i_\xi \mathbf{L} = -\int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \lambda \,, \tag{5.6}$$

where the minus sign arises due to our choice of orientation (see footnote 10). Hence,

$$\begin{aligned} \int_{\Sigma_{r_o}\cap C} \left(\mathbf{Q}[\xi] - i_\xi \mathbf{B}\right) &= \int_{\mathcal{H}} \mathbf{Q}[\xi] - \int_C i_\xi \mathbf{L} - \frac{1}{\kappa^2} \int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \left(K_{(d)} + \lambda_{\mathrm{ct}}\right) \\ &= \int_{\mathcal{H}} \mathbf{Q}[\xi] - \frac{1}{\kappa^2} \int_{\Sigma_{r_o}} d\sigma_i \xi^i \left(K_{(d)} - \lambda_{(d)}\right) . \end{aligned} \tag{5.7}$$

For $\xi^t = 0$ the last term vanishes. If however $\xi^t = 1$, then we can use the fact that the background is stationary to obtain

$$\frac{\beta}{\kappa^2} \int_{\partial\mathcal{M}\cap C} d\sigma_i \xi^i \left(K_{(d)} - \lambda_{(d)}\right) = -\frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^d x \sqrt{\gamma_E} \left(K_{(d)} - \lambda_{(d)}\right) \equiv I \,, \tag{5.8}$$

where γ_{Eij} is the euclidean metric. This completes the proof. □

5.1 Black hole thermodynamics

This lemma, besides relating the conserved charges to local integrals over the horizon, leads immediately to the quantum statistical relation [38]

$$I = \beta G\left(T, \Omega_i, \Phi\right) , \tag{5.9}$$

[12] Note that the integrals over $\mathcal{H}$ should be done with an *inward*-pointing unit vector.

[13] We assume throughout this paper that all fields are regular outside and on the horizon so that the application of Stokes' theorem is legitimate.

where

$$G\,(T,\Omega_i,\Phi_i) \equiv M - TS - \Omega_i J_i - \Phi\, Q\,, \tag{5.10}$$

is the Gibbs free energy. (5.9) follows trivially from lemma 2 provided

$$\mathcal{Q}[\chi] + \int_{\mathcal{H}} \mathbf{Q}[\chi] = M - TS - \Omega_i J_i - \Phi\, Q\,, \tag{5.11}$$

where χ is the null generator of the horizon, normalized such that $\chi^t = 1$. To show that this is the case we need a precise definition of the thermodynamic variables appearing in the Gibbs free energy.

Electric charge. Using Stokes' theorem, the electric charge[14] (4.7) is also given by

$$Q \equiv -\int_{\partial\mathcal{M}\cap C} *\mathcal{F} = -\int_{\mathcal{H}} *\mathcal{F}\,. \tag{5.12}$$

Electric potential. We define the electric potential, Φ, conjugate to the charge Q, by

$$\Phi \equiv -A_\mu \chi^\mu|_{\mathcal{H}}\,. \tag{5.13}$$

This is well-defined, for $A_\mu\chi^\mu$ is constant on $\mathcal{H}$. To see this consider a vector field t tangent to the horizon. Then,

$$t\cdot\partial(A_\mu\chi^\mu) = t^\rho(\chi^\mu F_{\rho\mu} + \mathcal{L}_\chi A_\rho) = t^\rho\chi^\mu F_{\rho\mu}\,. \tag{5.14}$$

But since t is tangent to $\mathcal{H}$, $t|_{\mathcal{H}} \propto \chi$ and hence

$$t\cdot\partial(A_\mu\chi^\mu)|_{\mathcal{H}} = 0\,. \tag{5.15}$$

Mass. In order to define the mass we have to supply an asymptotic timelike Killing vector. In contrast to asymptotically flat spacetimes, in AlAdS spacetimes there is an additional subtlety in that there can be a non-zero angular velocity, Ω_i^∞, at spatial infinity. This is the case, for example, for the Kerr-AdS black holes in Boyer-Lindquist coordinates as we will see below. In such a rotating frame, there are many timelike Killing vectors obtained by appropriate linear combinations of ∂_t and ∂_{ϕ_i}. Using a general timelike Killing vector will result in a conserved quantity that is a linear combination of the true mass and the angular momenta. To resolve this issue we first go to a non-rotating frame by the coordinate transformation

$$t' = t\,, \qquad \phi_i' = \phi_i - \Omega_i^\infty t\,. \tag{5.16}$$

In this frame there is no such ambiguity and one can define the mass, as usual, using the Killing vector $\partial_{t'}$. In terms of the original coordinates we have

$$\partial_{t'} = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial\phi_i}{\partial t'}\partial_{\phi_i} = \partial_t + \Omega_i^\infty\partial_{\phi_i}\,. \tag{5.17}$$

Therefore, the mass is defined as

$$M \equiv \mathcal{Q}[\partial_t + \Omega_i^\infty\partial_{\phi_i}]\,. \tag{5.18}$$

[14]We assume in this paper that the black holes are only electrically charged. If there are magnetic charges as well, one has to be careful with global issues.

Angular velocities. Let $\chi = \partial_t + \Omega_i^H \partial_{\phi_i}$ be the null generator of the horizon. This defines the angular velocities, Ω_i^H, of the horizon. We define the angular velocities, Ω_i, by

$$\Omega_i \equiv \Omega_i^H - \Omega_i^\infty \,. \tag{5.19}$$

Angular momenta. We define the angular momenta, J_i, by

$$J_i \equiv -\mathcal{Q}[\partial_{\phi_i}] = \int_{\mathcal{H}} \mathbf{Q}[\partial_{\phi_i}] \,, \tag{5.20}$$

where the second equality follows from lemma 2.

Entropy. Finally, using Wald's definition of the entropy [51] (see also [52]) we get

$$-\beta \int_{\mathcal{H}} \mathbf{Q}[\chi] = S + \beta \Phi \, Q \,. \tag{5.21}$$

With these definitions it is now straightforward to see that (5.11), and hence (5.9) hold.

5.2 First law

To derive the first law we consider variations that satisfy our boundary conditions. Namely, if the anomaly vanishes, then the boundary conditions (3.44) should be satisfied, otherwise (3.51) should hold, i.e. a representative of the conformal class should be kept fixed. We will discuss the significance of this in the next section. In other words, we only vary the normalizable mode of the solutions, as one might have anticipated on physical grounds.[15]

We now show, following Wald et al. [51, 53], that these variations satisfy the first law. From equation (D.8) we have

$$\mathrm{d}\left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) = \omega(\psi, \delta\psi, \mathcal{L}_\chi \psi) \,. \tag{5.22}$$

Hence,

$$\begin{aligned} \int_C \mathrm{d}\left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) &= \int_{\partial\mathcal{M}\cap C} \left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) - \int_{\mathcal{H}} \left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) \\ &= \int_C \omega(\psi, \delta\psi, \mathcal{L}_\chi \psi) = 0 \,, \end{aligned} \tag{5.23}$$

since χ is a Killing vector. However, χ is tangent to $\mathcal{H}$ and so we arrive at

$$\int_{\partial\mathcal{M}\cap C} \left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) = \int_{\mathcal{H}} \delta \mathbf{Q}[\chi] \,. \tag{5.24}$$

Consider first the left hand side. Writing $\chi = \partial_t + \Omega_i^\infty \partial_{\phi_i} + \Omega_i \partial_{\phi_i}$ and using the fact that ∂_{ϕ_i} is tangent to $\partial\mathcal{M}$ we get

$$\int_{\partial\mathcal{M}\cap C} \left(\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}\right) = \int_{\partial\mathcal{M}\cap C} \left(\delta \mathbf{Q}[\partial_t + \Omega_i^\infty \partial_{\phi_i}] - i_t \mathbf{\Theta}\right) + \Omega_i \int_{\partial\mathcal{M}\cap C} \delta \mathbf{Q}[\partial_{\phi_i}]$$

[15] Note that the non-normalizable mode determines the conformal class at the boundary. The non-normalizable mode *together* with a defining function specify a representative of the conformal class.

$$= \delta \int_{\partial\mathcal{M}\cap C} (\mathbf{Q}[\partial_t + \Omega_i^\infty \partial_{\phi_i}] - i_t \mathbf{B}) + \Omega_i \delta \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\partial_{\phi_i}]$$
$$= -(\delta M - \Omega_i \delta J_i)\,. \tag{5.25}$$

In order to evaluate the right hand side of (5.24) we need to match the horizons of the perturbed and unperturbed solutions [51], the unit surface gravity generators, $\tilde{\chi} \equiv \frac{1}{\tilde{\kappa}}\chi$, of the horizons and the electric potentials. From (5.21) then we immediately get

$$-\int_{\mathcal{H}} \delta\mathbf{Q}[\chi] = T\delta S + \Phi\delta Q\,. \tag{5.26}$$

Therefore, (5.24) is a statement of the first law, namely

$$\delta M = T\delta S + \Omega_i \delta J_i + \Phi \delta Q\,. \tag{5.27}$$

However, we emphasize that the variations in this expression must satisfy the appropriate boundary conditions that make the variational problem well-defined. Namely, if the anomaly vanishes, then the boundary conditions (3.44) should be satisfied, but if there is a non-zero anomaly, then (3.51) must be satisfied instead, i.e. the representative of the conformal class should be kept fixed. We will discuss the significance of this in the next section.

5.3 Dependence on the representative of the conformal class

Let us now discuss how the thermodynamic variables defined above depend on the representative of the conformal class at the boundary.

To this end we recall that a Weyl transformation on the boundary is induced by a PBH transformation, i.e. a combined bulk diffeomorphism and a compensating gauge transformation, given by (3.29) after setting $\xi_o = 0$ and $\alpha_o = 0$. However, in order to be able to compare the mass and angular momenta for the two representatives of the conformal class we require that the two representatives have the same maximal set of commuting isometries, i.e. we restrict to Weyl factors $\delta\sigma$ which are independent of the coordinates t, ϕ_α adapted to the asymptotic isometries.

It is now straightforward to see that all intensive thermodynamic variables, namely the temperature T, the angular velocities Ω_i and the electric potential Φ are invariant under such diffeomorphisms. The same holds for the entropy S, the angular momenta J_i, and the electric charge Q, since, as we saw above, they can be expressed as local integrals over the horizon. Therefore, the only quantities which could potentially transform non-trivially are the mass M and the on-shell euclidean action I. However, their transformations are not independent since they are constrained by the quantum statistical relation (5.9), namely

$$\delta_\sigma I = \beta \delta_\sigma M\,. \tag{5.28}$$

This is a significant result which cannot be seen easily otherwise. We know that

$$\delta_\sigma I = -\int_{\partial\mathcal{M}} d^d x \sqrt{\gamma_E}\mathcal{A}\delta\sigma\,, \tag{5.29}$$

while

$$\delta_\sigma M = -2\int_{\partial\mathcal{M}\cap C} d\sigma_i\{(2\tilde{\pi}_{(d)}{}^i_j + \tilde{\pi}_{(d)}{}^i A_j)\tilde{k}^j\delta\sigma + \cdots\}\,, \tag{5.30}$$

where $\tilde{k} = \partial_t + \Omega_i^\infty\partial_{\phi_i}$ and the dots stand for terms involving derivatives of the Weyl factor $\delta\sigma$. One can check this explicitly in certain examples by directly evaluating the transformation of the renormalized stress tensor under a PBH transformation [18, 19].

As a final point let us consider how (5.27) would be modified if there is a non-vanishing anomaly and we allow for variations which keep fixed only the conformal class and not a particular representative. In this case, (4.41) implies that

$$-\int_{\partial\mathcal{M}\cap C}(\delta\mathbf{Q}[\chi] - i_\chi\mathbf{\Theta}) = -T\delta_\sigma I + \delta M - \Omega_i\delta J_i\,, \tag{5.31}$$

and the first law should be modified to

$$\begin{aligned}\delta M &= T\delta_\sigma I + T\delta S + \Omega_i\delta J_i + \Phi\delta Q\\ &= \delta_\sigma M + T\delta S + \Omega_i\delta J_i + \Phi\delta Q\,,\end{aligned} \tag{5.32}$$

where $\delta\sigma$ is the Weyl factor by which the representative of the conformal class is changed due to the variation δ and the second equality follows from (5.28).

We can now state precisely how the first law works in the presence of a non-vanishing anomaly. A generic variation δ will not keep the conformal representative fixed and it will induce a Weyl transformation $\delta\sigma$. We should then undo this Weyl transformation by a PBH transformation with Weyl factor $-\delta\sigma$. Then, (5.32) ensures that the combined variation, which *does* keep the conformal representative fixed, satisfies the usual first law. The general Kerr-AdS black hole in five dimensions provides a clear illustration of this.

6. Examples

In this section we will demonstrate our analysis by two examples, the Kerr-Newman-AdS black hole in four dimensions [60, 61] and the general Kerr-AdS black hole in five dimensions [55]. The second example provides a clear illustration of the role of the conformal anomaly in the thermodynamics.

Before focusing on the specific examples however we discuss the steps and subtleties involved in the computation. Recall that the defining feature of the counterterm method is that the on-shell action of AdS gravity can be rendered finite on *any* solution by adding to the action a set of *local covariant* boundary counterterms. One should not forget, however, that the precise form of the counterterms depends on the regularization/renormalization scheme. The counterterms used in the literature were derived using as regulator a cut-off in the Fefferman-Graham radial coordinate z [16], or equivalently in the radial coordinate r we use in this paper. The cut-off hypersurface $r = r_o$ is in general different from the hypersurfaces $\tilde{r} = const.$, where $\tilde{r}$ is another radial coordinate that might appear naturally in the bulk metric. So, to evaluate correctly the counterterm contribution to the on-shell action, one should transform asymptotically the solution to Fefferman-Graham coordinates and then evaluate the counterterm action (or equivalently transform the hypersurface $r =$

r_o and the counterterm action in the new coordinates). Of course, it is always possible to work with a different regulator but then the counterterm action should be worked out from scratch.

Let us discuss now the evaluation of the conserved charges. Given that the asymptotics and counterterms are universal, one can work out in full generality the explicit form of the renormalized stress energy tensor in terms of the metric coefficients $\mathsf{g}_{(m)}$ that appear in the asymptotic expansion of the solutions of a given action. This is done for pure gravity in [18] and for gravity coupled to certain matter in [28]. To evaluate the holographic stress tensor on a specific solution one could thus simply read off the metric coefficients from the asymptotic expansion of the metric and plug them in the general formula. This is the simplest way to proceed if the explicit expression for the holographic stress energy tensor is known. If this is not the case, it is simpler to just compute from the asymptotics of the given solution the contribution of the bulk and counterterm actions to the holographic stress energy tensor and add them up to produce a finite answer. To evaluate the conserved charges we finally integrate the holographic stress energy tensor contracted with the appropriate asymptotic conformal Killing vector over the appropriate domain. The only remaining subtlety is the choice of a timelike Killing vector to be used in the definition of mass when the boundary metric is in a rotating frame. In this case we choose the Killing vector that corresponds to the standard timelike Killing vector $\partial/\partial t$ is the corresponding non-rotating frame.

Below we describe our calculation for the four dimensional Kerr-Newman-AdS black hole in considerable detail, mainly in order to emphasize the role of the Fefferman-Graham coordinate system in the asymptotic analysis, which is not fully appreciated in the literature. We then turn to the five dimensional Kerr-AdS black hole, emphasizing the role of the anomaly and its relation to the Casimir energy. Previous work on the thermodynamics of these black holes includes [2, 54−59, 21, 11, 12, 46].

6.1 $D = 4$ Kerr-Newman-AdS black hole

The metric of the Kerr-Newman-AdS black hole in Boyer-Lindquist coordinates reads [60, 61, 58]

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2, \quad (6.1)$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta\,, \\ \Delta_r &= (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr + q^2\,, \\ \Delta_\theta &= 1 - \frac{a^2}{l^2}\cos^2\theta\,, \qquad \Xi = 1 - \frac{a^2}{l^2}\,. \end{aligned} \quad (6.2)$$

The gauge potential in this coordinate system is given by

$$\mathbf{A} = -\frac{2qr}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right). \quad (6.3)$$

This metric and gauge field solve the Einstein-Maxwell equations which follow from the action (omitting the boundary terms)

$$I_{\text{lorentzian}} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F^2\right). \tag{6.4}$$

The event horizon is located at $r = r_+$, where r_+ is the largest root of $\Delta_r = 0$, and its area is

$$A = \frac{4\pi(r_+^2 + a^2)}{\Xi}. \tag{6.5}$$

The analytic continuation of the lorentzian metric (6.1) by $t = -i\tau$, $a = i\alpha$ develops a conical singularity unless we periodically identify $\tau \sim \tau + \beta$ and $\phi \sim \phi + i\beta\Omega_H$, where

$$\beta = \frac{4\pi(r_+^2 + a^2)}{r_+\left(1 + \frac{a^2}{l^2} + \frac{3r_+^2}{l^2} - \frac{a^2+q^2}{r_+^2}\right)}, \tag{6.6}$$

is the inverse temperature and the angular velocity of the horizon, Ω_H, is given by

$$\Omega_H = \frac{a\Xi}{r_+^2 + a^2}. \tag{6.7}$$

However, in this coordinate system there is a non-zero angular velocity at infinity, namely

$$\Omega_\infty = -\frac{a}{l^2}. \tag{6.8}$$

Following our prescription (5.19), we define the angular velocity relevant for the thermodynamics as the difference (see also [58, 21])

$$\Omega = \Omega_H - \Omega_\infty = \frac{a(1 + r_+/l^2)}{r_+^2 + a^2}. \tag{6.9}$$

Finally, if $\chi = \partial_t + \Omega_H\partial_\phi$ is the null generator of the Killing horizon, the electric potential is given by

$$\Phi \equiv -A_\mu\chi^\mu|_{r_+} = \frac{2qr_+}{r_+^2 + a^2}. \tag{6.10}$$

Next we determine the electric charge, angular momentum and mass, as well as the euclidean on-shell action of the Kerr-Newman-AdS solution. Our general analysis of the charges in section 4 showed that the counterterms do not contribute to the value of the electric charge or the angular momentum (lemma 1). However, the counterterms are essential for evaluating the mass and the on-shell action. Starting with the electric charge we easily find

$$Q \equiv -\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}\cap C} *\mathrm{d}\mathbf{A} = \frac{4\pi q}{\kappa^2\Xi}. \tag{6.11}$$

The angular momentum can be evaluated equally easily as

$$J \equiv \int_{\partial\mathcal{M}\cap C} \mathbf{Q}[\partial_\phi] = \frac{8\pi m a}{\kappa^2\Xi^2}. \tag{6.12}$$

Before we can calculate the mass and the on-shell euclidean action, we must first carry out the asymptotic analysis and determine the counterterms. Expanding the metric (6.1) for large r we get

$$\begin{aligned} ds^2 =& -\frac{r^2}{l^2}\left[1+\left(1+\frac{a^2}{l^2}\sin^2\theta\right)\frac{l^2}{r^2}-\frac{2ml^2}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\right]\left(dt-\frac{a\sin^2\theta}{\Xi}d\phi\right)^2+ \\ &+\frac{l^2}{r^2}\left[1-\left(1+\frac{a^2}{l^2}\sin^2\theta\right)\frac{l^2}{r^2}+\frac{2ml^2}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\right]dr^2+ \\ &+\frac{r^2}{\Delta_\theta}\left(1+\frac{a^2}{r^2}\cos^2\theta\right)d\theta^2+ \\ &+\frac{r^2\Delta_\theta\sin^2\theta}{\Xi^2}\left[d\phi^2+\frac{a^2}{r^2}\left((1+\sin^2\theta)d\phi^2-\frac{2\Xi}{a}d\phi dt\right)+\mathcal{O}\left(\frac{1}{r^4}\right)\right]. \end{aligned} \tag{6.13}$$

This metric is not of the standard form since the coefficient of the radial line element depends on the angle θ. Indeed the standard counterterms are derived using a Fefferman-Graham coordinate system of the form (A.1) [16, 18, 25, 29]. These counterterms, defined on hypersurfaces of *constant Fefferman-Graham radial coordinate*, are not necessarily the correct counterterms on the hypersurfaces of constant Boyer-Lindquist radial coordinate, as is widely assumed in the literature. Of course, it is in principle possible to choose a gauge which asymptotes to the Boyer-Lindquist form of the Kerr-AdS metric and carry out the asymptotic analysis from scratch using a regulator of constant Boyer-Lindquist radial coordinate and rederive the appropriate counterterms for such a regulator. However, it is much more efficient to bring the metric (6.13) into the Fefferman-Graham form and use the standard counterterms.

To this end we introduce new coordinates

$$\begin{aligned} \bar{r} &= r+\frac{1}{r}f(\theta)+\mathcal{O}\left(\frac{1}{r^3}\right), \\ \bar{\theta} &= \theta+\frac{1}{r^4}h(\theta)+\mathcal{O}\left(\frac{1}{r^6}\right), \end{aligned} \tag{6.14}$$

or

$$\begin{aligned} r &= \bar{r}\left[1-\frac{1}{\bar{r}^2}f(\bar{\theta})+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\ \theta &= \bar{\theta}-\frac{1}{\bar{r}^4}h(\bar{\theta})+\mathcal{O}\left(\frac{1}{\bar{r}^6}\right). \end{aligned} \tag{6.15}$$

Requiring that the coefficient of the new radial line element has no angular dependence and that there is no mixed term $d\bar{r}d\bar{\theta}$ in the metric uniquely fixes the functions $f(\bar{\theta})$ and $h(\bar{\theta})$ to be

$$\begin{aligned} f(\bar{\theta}) &= -\frac{a^2}{4}\cos^2\bar{\theta}, \\ h(\bar{\theta}) &= \frac{1}{8}l^2a^2\Delta_{\bar{\theta}}\sin\bar{\theta}\cos\bar{\theta}. \end{aligned} \tag{6.16}$$

In the new coordinate system the asymptotic form of the metric (6.1) becomes

$$\begin{aligned} ds^2 = & \frac{l^2}{\bar{r}^2}\left[1-\left(1+\frac{a^2}{l^2}\right)\frac{l^2}{\bar{r}^2}+\frac{2ml^2}{\bar{r}^3}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]d\bar{r}^2+ \\ & +\frac{\bar{r}^2}{\Delta_{\bar{\theta}}}\left[1+\frac{3}{2}\frac{a^2}{\bar{r}^2}\cos^2\bar{\theta}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]d\bar{\theta}^2- \\ & -\frac{\bar{r}^2}{l^2}\left[1+\left(1+\frac{a^2}{l^2}-\frac{a^2}{2l^2}\cos^2\bar{\theta}\right)\frac{l^2}{\bar{r}^2}-\frac{2ml^2}{\bar{r}^3}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]\left(dt-\frac{a\sin^2\bar{\theta}}{\Xi}d\phi\right)^2+ \\ & +\frac{\bar{r}^2\Delta_{\bar{\theta}}\sin^2\bar{\theta}}{\Xi^2}\left[d\phi^2+\frac{a^2}{\bar{r}^2}\left((2-\frac{1}{2}\cos^2\bar{\theta})d\phi^2-\frac{2\Xi}{a}d\phi dt\right)+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \end{aligned} \tag{6.17}$$

which is almost of the desired form. For later convenience let us write explicitly the components of the induced metric:

$$\begin{aligned} \gamma_{\bar{\theta}\bar{\theta}} &= \frac{\bar{r}^2}{\Delta_{\bar{\theta}}}\left[1+\frac{3}{2}\frac{a^2}{\bar{r}^2}\cos^2\bar{\theta}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\ \gamma_{tt} &= -\frac{\bar{r}^2}{l^2}\left[1+\left(1+\frac{a^2}{l^2}-\frac{a^2}{2l^2}\cos^2\bar{\theta}\right)\frac{l^2}{\bar{r}^2}-\frac{2ml^2}{\bar{r}^3}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\ \gamma_{t\phi} &= \frac{\bar{r}^2 a\sin^2\bar{\theta}}{l^2\Xi}\left[1+\left(1+\frac{1}{2}\cos^2\bar{\theta}\right)\frac{a^2}{\bar{r}^2}-\frac{2ml^2}{\bar{r}^3}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right], \\ \gamma_{\phi\phi} &= \frac{\bar{r}^2\sin^2\bar{\theta}}{\Xi}\left[1+\left(1+\frac{1}{2}\cos^2\bar{\theta}\right)\frac{a^2}{\bar{r}^2}+\frac{2ma^2\sin^2\bar{\theta}}{\bar{r}^3\Xi}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]. \end{aligned} \tag{6.18}$$

We can now introduce a cut-off at $\bar{r}=\bar{r}_o$ and proceed with the asymptotic analysis in the standard fashion. Note that the regulating surface $\bar{r}=\bar{r}_o$ becomes angle-dependent in the Boyer-Lindquist coordinates, namely

$$r_o(\theta)=\bar{r}_o\left[1+\frac{a^2}{4\bar{r}_o^2}\cos^2\theta+\mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right)\right]. \tag{6.19}$$

This is precisely the reason why the counterterms on a regulating surface defined by $r_o=$ constant are not necessarily the same as the counterterms on a regulating surface defined by $\bar{r}_o=$ constant.

Finally, to bring the metric in the form (A.1) we define the canonical radial coordinate

$$dr_*=l\left[1-\frac{1}{2}\left(1+\frac{a^2}{l^2}\right)\frac{l^2}{\bar{r}^2}+\frac{ml^2}{\bar{r}^3}+\mathcal{O}\left(\frac{1}{\bar{r}^4}\right)\right]\frac{d\bar{r}}{\bar{r}}. \tag{6.20}$$

Counterterms.[16] Following the standard algorithm for the asymptotic analysis we find that the counterterm action for the Maxwell-AdS gravity system in four dimensions is

$$I_{\rm ct}=\frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}}d^3x\sqrt{\gamma_E}\left(\frac{2}{l}+\frac{l}{2}R\right). \tag{6.21}$$

[16]We give the counterterms for the euclidean action which we want to evaluate. The counterterms for the lorentzian action are easily obtained by analytic continuation.

On-shell euclidean action. We are now ready to evaluate the renormalized on-shell euclidean action

$$I_{\rm ren} = -\frac{1}{2\kappa^2}\int_{\mathcal{M}_{\bar{r}_o}} d^4x\sqrt{g_E}\left(R[g_E]+\frac{6}{l^2}-\frac{1}{4}F^2\right) - \frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^3x\sqrt{\gamma_E}\left(K-\frac{2}{l}-\frac{l}{2}R\right). \tag{6.22}$$

Since the background is stationary, the bulk integral gives

$$\begin{aligned}&\frac{\beta}{2\kappa^2}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_{r_+}^{r_o(\theta)}dr\sqrt{g_E}\left(\frac{6}{l^2}+\frac{1}{4}F^2\right)=\\ &\quad=\frac{4\pi\beta}{\kappa^2 l^2\Xi}\left[\bar{r}_o\left(\bar{r}_o^2+\frac{5}{4}a^2\right)-r_+(r_+^2+a^2)-\frac{q^2l^2r_+}{r_+^2+a^2}+\mathcal{O}\left(\frac{1}{\bar{r}_o}\right)\right].\end{aligned} \tag{6.23}$$

Moreover, the boundary term is

$$-\frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^3x\sqrt{\gamma_E}\left(K-\frac{2}{l}-\frac{l}{2}R\right) = -\frac{4\pi\beta}{\kappa^2l^2\Xi}\left[\bar{r}_o(\bar{r}_o^2+\frac{5}{4}a^2)-ml^2+\mathcal{O}\left(\frac{1}{\bar{r}_o}\right)\right]. \tag{6.24}$$

Hence,

$$I_{\rm ren} = \frac{4\pi\beta}{\kappa^2l^2\Xi}\left[ml^2-r_+(r_+^2+a^2)-\frac{q^2l^2r_+}{r_+^2+a^2}\right]. \tag{6.25}$$

Renormalized stress tensor and conserved charges. We need now to evaluate the renormalized stress tensor

$$T_{(3)}{}^i_j = -\frac{l}{\kappa^2}\left(K_{(3)}{}^i_j - K_{(3)}\delta^i_j\right). \tag{6.26}$$

This can be done either by first writing the renormalized stress tensor in terms of the coefficients in the Fefferman-Graham expansion of the metric [18, 28, 29] and then reading off the coefficients from (6.18), or by first evaluating the extrinsic curvature using

$$K_{ij} = \frac{1}{2}\frac{d\bar{r}}{dr_*}\frac{d}{d\bar{r}}\gamma_{ij}\,, \tag{6.27}$$

and then subtracting the appropriate counterterms, namely

$$T_{(3)}{}^i_j = -\frac{l}{\kappa^2}\left(K^i_j - K\delta^i_j + \frac{2}{l}\delta^i_j - lR^i_j + \frac{1}{2}lR\delta^i_j\right) + \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right). \tag{6.28}$$

In any case we find (in agreement with [58])

$$\begin{aligned}T_{(3)}{}^t_t &= -\frac{2m}{\kappa^2}\frac{l^3}{\bar{r}_o^3}+\mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right),\\ T_{(3)}{}^{\bar\theta}_{\bar\theta} &= T_{(3)}{}^\phi_\phi = \frac{m}{\kappa^2}\frac{l^3}{\bar{r}_o^3}+\mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right),\\ T_{(3)}{}^t_\phi &= \frac{3m}{\kappa^2}\frac{a\sin^2\bar\theta}{l\Xi}\frac{l^3}{\bar{r}_o^3}+\mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right),\\ T_{(3)}{}^\phi_t &= \mathcal{O}\left(\frac{1}{\bar{r}_o^4}\right).\end{aligned} \tag{6.29}$$

For this solution one can easily show that the gauge field momentum does not contribute to the holographic charge (4.28) and so, for any boundary conformal Killing vector, ξ, we have

$$\mathcal{Q}[\xi] = -\int_0^{2\pi} d\phi \int_0^{\pi} d\bar{\theta}\sqrt{-\gamma}T_{(3)}{}^t_j\xi^j \,. \tag{6.30}$$

As a check, we evaluate

$$\mathcal{Q}[-\partial_\phi] = \frac{8\pi ma}{\kappa^2\Xi^2} \,, \tag{6.31}$$

in complete agreement with (6.12).

To obtain the mass now we first need to identify the correct timelike Killing vector. This can be done unambiguously as follows. From the asymptotic form of the metric in Boyer-Lindquist coordinates we see that the corresponding boundary metric is not the standard metric on $\mathbb{R}\times\mathbf{S}^2$ even for $m = q = 0$, since there is a non-zero angular velocity $\Omega_\infty = -\frac{a}{l^2}$. However, as it is discussed e.g. in [21], this boundary metric is *conformal* to the standard boundary metric of AdS_4. To see this we perform a coordinate transformation from the coordinates $(t, \bar{\theta}, \phi)$ to $(t', \bar{\theta}', \phi')$, given by

$$t' = t \,, \qquad \phi' = \phi + \frac{a}{l^2}t \,, \qquad \Xi\tan^2\bar{\theta}' = \tan^2\bar{\theta} \,. \tag{6.32}$$

The resulting boundary metric in the new coordinates is the standard metric on $\mathbb{R}\times\mathbf{S}^2$ up to the conformal factor $\cos^2\bar{\theta}/\cos^2\bar{\theta}'$. It follows that the correct timelike Killing vector that defines the mass is

$$\partial_{t'} = \frac{\partial t}{\partial t'}\partial_t + \frac{\partial\phi}{\partial t'}\partial_\phi = \partial_t - \frac{a}{l^2}\partial_\phi \,, \tag{6.33}$$

in agreement with (5.17). Hence,

$$M \equiv \mathcal{Q}[\partial_t - \frac{a}{l^2}\partial_\phi] = \frac{8\pi m}{\kappa^2\Xi^2} \,. \tag{6.34}$$

This is precisely the mass obtained in [21] by integrating the first law.[17] Finally, defining the entropy by

$$S = \frac{2\pi}{\kappa^2}A \,, \tag{6.35}$$

it can now be easily seen that the quantum statistical relation (5.9) as well as the first law (5.27) are satisfied.

6.2 $D = 5$ Kerr-AdS black hole

As a second example we consider the general five dimensional Kerr-AdS solution [55], which illustrates the role of the conformal anomaly.

[17]Note that our timelike Killing vector is *different* from the Killing vector, $\partial_t + \frac{a}{l^2}\partial_\phi$, which the authors of [21] claim makes the conformal mass [1, 50, 56] equal to the mass obtained from the first law.

In Boyer-Lindquist coordinates the metric is[18]

$$
\begin{aligned}
ds^2 =& -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi_a}d\phi\right)^2 + \\
&+\frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(bdt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \\
&+\frac{(1+r^2l^{-2})}{r^2\rho^2}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2,
\end{aligned}
\tag{6.36}
$$

where

$$
\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \\
\Delta_r &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)\left(1+\frac{r^2}{l^2}\right) - 2m\,, \\
\Delta_\theta &= 1 - \frac{a^2}{l^2}\cos^2\theta - \frac{b^2}{l^2}\sin^2\theta\,, \\
\Xi_a &= 1 - \frac{a^2}{l^2}\,, \qquad \Xi_b = 1 - \frac{b^2}{l^2}\,.
\end{aligned}
\tag{6.37}
$$

The event horizon is located at $r = r_+$, where r_+ is the largest root of $\Delta_r = 0$, and its area is

$$
A = \frac{2\pi^2(r_+^2+a^2)(r_+^2+b^2)}{r_+\Xi_a\Xi_b}\,.
\tag{6.38}
$$

The inverse temperature is given by

$$
\beta = 2\pi\left[r_+\left(1+\frac{r_+^2}{l^2}\right)\left(\frac{1}{r_+^2+a^2}+\frac{1}{r_+^2+b^2}\right) - \frac{1}{r_+}\right]^{-1}.
\tag{6.39}
$$

The angular velocities relative to a non-rotating frame at infinity are

$$
\Omega_a = \frac{a(1+r_+^2l^{-2})}{r_+^2+a^2}\,, \qquad \Omega_b = \frac{b(1+r_+^2l^{-2})}{r_+^2+b^2}\,,
\tag{6.40}
$$

and the corresponding angular momenta are easily evaluated

$$
J_a = \int_{\partial\mathcal{M}\cap C}\mathbf{Q}[\partial_\phi] = \frac{4\pi^2 ma}{\kappa^2\Xi_a^2\Xi_b}\,,
\tag{6.41}
$$

$$
J_b = \int_{\partial\mathcal{M}\cap C}\mathbf{Q}[\partial_\psi] = \frac{4\pi^2 mb}{\kappa^2\Xi_a\Xi_b^2}\,.
\tag{6.42}
$$

As for the four-dimensional Kerr-Newman-AdS black hole, in order to bring the metric into the standard asymptotic form, we need to introduce the new coordinates

$$
\begin{aligned}
r &= \bar r\left\{1 + \frac{1}{4}\hat\Delta_{\bar\theta}\frac{l^2}{\bar r^2} + \frac{1}{16}\hat\Delta_{\bar\theta}(1+\hat\Xi_a+\hat\Xi_b-2\hat\Delta_{\bar\theta})\frac{l^4}{\bar r^4} + \mathcal{O}\left(\frac{1}{\bar r^6}\right)\right\}, \qquad (6.43)\\
\theta &= \bar\theta + \frac{1}{16}(1-\hat\Delta_{\bar\theta})\hat\Delta'_{\bar\theta}\frac{l^4}{\bar r^4} - \frac{1}{32}(1-\hat\Delta_{\bar\theta})\hat\Delta'_{\bar\theta}(1+\hat\Xi_a+\hat\Xi_b+3\hat\Delta_{\bar\theta})\frac{l^6}{\bar r^6} + \mathcal{O}\left(\frac{1}{\bar r^8}\right),
\end{aligned}
$$

[18] Note that $0 \le \theta \le \pi/2$ in five dimensions, while $0 \le \theta \le \pi$ in four dimensions.

where, to simplify the notation, we have defined

$$\hat{\Delta}_\theta = 1 - \Delta_\theta\,, \qquad \hat{\Xi}_a = 1 - \Xi_a\,, \qquad \hat{\Xi}_b = 1 - \Xi_b\,. \tag{6.44}$$

In the new coordinate system the induced metric, up to terms of order $1/\bar{r}^6$ inside the braces, takes the form

$$\begin{aligned}
\gamma_{\bar\theta\bar\theta} &= \frac{\bar{r}^2}{\Delta_{\bar\theta}}\left\{1 + \frac{3\hat{\Delta}_{\bar\theta}}{2}\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[\left(1 - \frac{3\hat{\Delta}_{\bar\theta}}{2}\right)\left(\hat{\Xi}_a + \hat{\Xi}_b - \frac{3\hat{\Delta}_{\bar\theta}}{2}\right) + \hat{\Xi}_a\hat{\Xi}_b\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{tt} &= -\frac{\bar{r}^2}{l^2}\left\{1 + \left(1 + \hat{\Xi}_a + \hat{\Xi}_b - \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\frac{l^2}{\bar{r}^2} + \left[\frac{\hat{\Delta}_{\bar\theta}}{8}\left(1 + \hat{\Xi}_a + \hat{\Xi}_b - \frac{3\hat{\Delta}_{\bar\theta}}{2}\right) - \frac{2m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{t\phi} &= \frac{\bar{r}^2}{l^2}\frac{a\sin^2\bar\theta}{\Xi_a}\left\{1 + \left(\hat{\Xi}_a + \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[\left(\hat{\Xi}_b - \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\left(\Xi_a - \frac{\hat{\Delta}_{\bar\theta}}{2}\right) + \hat{\Xi}_a\hat{\Xi}_b - \frac{8m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{t\psi} &= \frac{\bar{r}^2}{l^2}\frac{b\cos^2\bar\theta}{\Xi_b}\left\{1 + \left(\hat{\Xi}_b + \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\frac{l^2}{\bar{r}^2} + \frac{1}{4}\left[(\hat{\Xi}_a - \frac{\hat{\Delta}_{\bar\theta}}{2}\Big)\left(\Xi_b - \frac{\hat{\Delta}_{\bar\theta}}{2}\right) + \hat{\Xi}_a\hat{\Xi}_b - \frac{8m}{l^2}\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{\phi\phi} &= \bar{r}^2\frac{\sin^2\bar\theta}{\Xi_a}\left\{1 + \left(\hat{\Xi}_a + \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\frac{l^2}{\bar{r}^2} + \right.\\
&\qquad\qquad \left. + \frac{1}{4}\left[\left(\hat{\Xi}_b - \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\left(\Xi_a - \frac{\hat{\Delta}_{\bar\theta}}{2}\right) + \hat{\Xi}_a\hat{\Xi}_b + \frac{8m}{l^2}\frac{a^2\sin^2\bar\theta}{l^2\Xi_a}\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{\psi\psi} &= \bar{r}^2\frac{\cos^2\bar\theta}{\Xi_b}\left\{1 + \left(\hat{\Xi}_b + \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\frac{l^2}{\bar{r}^2} + \right.\\
&\qquad\qquad \left. + \frac{1}{4}\left[\left(\hat{\Xi}_a - \frac{\hat{\Delta}_{\bar\theta}}{2}\right)\left(\Xi_b - \frac{\hat{\Delta}_{\bar\theta}}{2}\right) + \hat{\Xi}_a\hat{\Xi}_b + \frac{8m}{l^2}\frac{b^2\cos^2\bar\theta}{l^2\Xi_b}\right]\frac{l^4}{\bar{r}^4}\right\},\\
\gamma_{\phi\psi} &= \bar{r}^2\left\{\frac{2m}{l^2}\frac{a\cos^2\bar\theta}{l\Xi_a}\frac{b\sin^2\bar\theta}{l\Xi_b}\frac{l^4}{\bar{r}^4}\right\},
\end{aligned} \tag{6.45}$$

while the canonical radial coordinate r_* is given by

$$dr_* = l\left\{1 - \frac{1}{2}(1 + \hat{\Xi}_a + \hat{\Xi}_b)\frac{l^2}{\bar{r}^2} + \left[\frac{m}{l^2} + \frac{1}{8}(1 + \hat{\Xi}_a + \hat{\Xi}_b)^2 + \frac{1}{4}(1 + \hat{\Xi}_a^2 + \hat{\Xi}_b^2)\right]\frac{l^4}{\bar{r}^4}\right\}\frac{d\bar{r}}{\bar{r}}\,. \tag{6.46}$$

On-shell euclidean action. The renormalized euclidean action in five dimensions is given by

$$\begin{aligned}
I_{\rm ren} = &-\frac{1}{2\kappa^2}\int_{\mathcal{M}_{\bar{r}_o}} d^5x\sqrt{g_E}\left(R[g_E] + \frac{12}{l^2}\right) - \\
&-\frac{1}{\kappa^2}\int_{\Sigma_{\bar{r}_o}} d^4x\sqrt{\gamma_E}\left(K - \frac{3}{l} - \frac{l}{4}R + \frac{l^3}{16}(R_{ij}R^{ij} - \frac{1}{3}R^2)\log e^{-2\bar{r}_o}\right).
\end{aligned} \tag{6.47}$$

Evaluating this expression we obtain

$$I = \beta M_{\rm Casimir} + \frac{2\pi^2\beta}{\kappa^2 l^2\Xi_a\Xi_b}\left[ml^2 - (r_+^2 + a^2)(r_+^2 + b^2)\right], \tag{6.48}$$

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where

$$M_{\text{Casimir}} \equiv \frac{3\pi^2 l^2}{4\kappa^2}\left(1+\frac{(\Xi_a-\Xi_b)^2}{9\Xi_a\Xi_b}\right). \tag{6.49}$$

This expression for the on-shell euclidean action is precisely the expression obtained in [59] and it differs from that of [21] by the term involving the Casimir energy. Moreover, (6.49) is equal the Casimir energy of the field theory on the rotating Einstein universe [59].

Evaluating the holographic mass we find

$$M \equiv \mathcal{Q}[\partial_t - \frac{a}{l^2}\partial_\phi - \frac{b}{l^2}\partial_\psi] = M_{\text{Casimir}} + \frac{2\pi^2 m(2\Xi_a+2\Xi_b-\Xi_a\Xi_b)}{\kappa^2\Xi_a^2\Xi_b^2}\,, \tag{6.50}$$

which again agrees with the mass obtained in [21] except for the Casimir energy part. However, except for the Casimir energy, this mass is not the same as the one given in [59]. The discrepancy arises presumably because [59] do not use the correct non-rotating timelike Killing vector to evaluate the mass.

With the expressions for the mass and on-shell action we have obtained, one can easily see that the quantum statistical relation (5.9) is satisfied, despite the presence of the Casimir energy. However, to show that our expressions do satisfy the first law, we need to examine the effect of an arbitrary variation of the parameters a, b and m on the representative of the conformal class at the boundary.

The boundary metric is

$$d\bar{s}^2 = -dt^2 + \frac{2a\sin^2\bar\theta}{\Xi_a}dtd\phi + \frac{2b\cos^2\bar\theta}{\Xi_b}dtd\psi + \frac{l^2}{\Delta_{\bar\theta}}d\bar\theta^2 + \frac{l^2\sin^2\bar\theta}{\Xi_a}d\phi^2 + \frac{l^2\cos^2\bar\theta}{\Xi_b}d\psi^2\,. \tag{6.51}$$

Under a variation of the angular parameters a, b, this metric is *not* kept fixed as is required by the variational problem. The conformal class however is kept fixed (up to a diffeomorphism). To see this first consider the variation of (6.51) w.r.t. a and b, and then perform the compensating infinitesimal diffeomorphism

$$t=t'\,,\quad \tan^2\bar\theta = \left(1+\frac{\delta\Xi_a}{\Xi_a}-\frac{\delta\Xi_b}{\Xi_b}\right)\tan^2\bar\theta'\,,\quad \phi=\phi'-\frac{\delta a}{l^2}t'\,,\quad \psi=\psi'-\frac{\delta b}{l^2}t'\,. \tag{6.52}$$

The result of the combined transformation is

$$d\bar{s}^2 \to \left(1-\frac{\delta\Xi_a}{\Xi_a}\sin^2\bar\theta - \frac{\delta\Xi_b}{\Xi_b}\cos^2\bar\theta\right)d\bar{s}^2\,. \tag{6.53}$$

The variation of the on-shell action due to this Weyl factor is

$$\delta_\sigma I = -\int_{\partial\mathcal{M}} d^dx\sqrt{\gamma_E}\mathcal{A}\delta\sigma = \frac{\pi^2\beta l^2}{12\kappa^2}\delta\left(\frac{\Xi_a}{\Xi_b}+\frac{\Xi_b}{\Xi_a}\right) = \beta\delta M_{\text{Casimir}} = \beta\delta_\sigma M\,, \tag{6.54}$$

where the last equality follows from (5.28). Therefore, as expected, only the Casimir energy part of the mass transforms non trivially under a Weyl transformation.

Summarizing, we have shown that under a generic variation of the parameters a, b and m

$$\delta M = \delta M_{\text{Casimir}} + T\delta S + \Omega_a \delta J_a + \Omega_b \delta J_b \,, \tag{6.55}$$

in complete agreement with (5.32). The first law then is satisfied once we accompany such a generic variation with a compensating PBH transformation which undoes the Weyl transformation of the representative of the conformal class.[19]

7. Conclusion

We discussed in this paper the variational problem for AdS gravity, the definition of conserved charges and the first law of thermodynamics for asymptotically locally AdS black hole spacetimes. We conclude by summarizing the main points.

AlAdS spacetimes are solutions of Einstein's equations whose Riemann tensor is asymptotically equal to the Riemann tensor of AdS but their asymptotic global structure is not necessarily that of AdS. Their metric tensor induces a conformal structure at infinity and so, a natural set of Dirichlet boundary conditions for AdS gravity is that a conformal structure is kept fixed. Notice that any other choice of Dirichlet boundary conditions would break part of the bulk diffeomorphisms, namely the ones that induce a Weyl transformation at the boundary. We examined the variational problem for such Dirichlet boundary conditions and found that it is well-posed provided the conformal anomaly $\mathcal{A}$ is zero and we add to the action (in addition to the Gibbons-Hawking term) a set of new boundary terms. These new boundary terms are precisely the boundary counterterms introduced in [16, 17] in order to achieve finiteness of the on-shell action and of the holographic stress energy tensor. If the conformal anomaly is non-zero, however, one has to choose a specific representative of the boundary conformal structure to make the variational problem well-posed, thus breaking part of the bulk diffeomorphisms. In this case the boundary counterterms guarantee that the on-shell action has a well-defined transformation under the broken diffeomorphisms, the transformation rule being determined by the conformal anomaly. In other words, we need to pick a reference representative in this case, but the charge from one representative to another is essentially determined by the conformal class of the boundary metric via the conformal anomaly.

We then derived the conserved charges for AlAdS spacetimes that possess asymptotic symmetries. The holographic charges were originally derived [17 − 19] using the Brown-York prescription [20] supplemented by appropriate boundary counterterms [16]. Here we derived the conserved charges using Noether's method and the covariant phase space method of Wald *et al* [51, 53, 24] and found that they are equal to the holographic charges.[20] Notice that the case of $AAdS_{2k+1}$ spacetimes, i.e. ones that approach asymptotically the exact AdS_{2k+1} solution, is special in that there exists a covariantly conserved stress energy tensor constructed locally from the boundary metric that can be used to off-set the charges

[19] Of course we should also perform a compensating diffeomorphism (6.52), but this does not affect the first law since all thermodynamic variables are invariant under such a diffeomorphism.

[20] The fact that the holographic charges are associated with asymptotic symmetries was also recently shown in [22] using somewhat different methods.

such that their value is equal to zero for the AdS_{2k+1} solution. This tensor is equal to the holographic stress energy tensor of AdS_{2k+1} [18]. When such off-set is done, i.e. when the charges are measured relative to AdS_{2k+1}, the conserved charges agree with those of [1, 2]. A detailed comparison between different notions of conserved charges for AAdS spacetimes was recently presented in [9].

We next considered general stationary, axisymmetric, charged AlAdS black holes in any dimension and showed in general that the quantum statistical relation (or Smarr formula) and the first law of thermodynamics hold. We would like to emphasize that the variations that enter in the first law need not respect any of the symmetries of the solution but they have to respect the boundary conditions. In other words, there are general normalizable variations keeping fixed the non-normalizable mode (see footnote 15). In some cases, such as the five dimensional Kerr-AdS solution in Boyer-Lindquist coordinates, the solution is parameterized such that the mass and other conserved charges depend on parameters that also appear in the boundary metric. When varying these parameters one varies not only the conserved charges but also the boundary metric, thus violating the boundary conditions. To keep fixed the non-normalizable mode one must perform a compensating coordinate transformation, and taking this into account one finds that the first law is satisfied, resolving a puzzle in the literature where it seemed that only the charges relative to AdS satisfy the first law [14].

We illustrated our discussion by computing the conserved quantities for the four dimensional Kerr-Newman-AdS and the five dimensional Kerr-AdS black hole. An important point to realize is that the usual counterterms are defined on the hypersurface $z = const.$, where z is the Fefferman-Graham coordinate. It is not in general correct to use the same set of counterterms when the cut-off hypersurface is different (chosen for instance by considering $r = const.$ surfaces, where r is a different radial coordinate that might appear naturally in the bulk solution). So, to correctly compute the contribution of the counterterms to the on-shell action one should asymptotically transform the bulk metric to the Fefferman-Graham coordinates. Another subtle point is about the choice of timelike Killing vector in the definition of mass when the boundary metric is in a rotating frame. In this case one can resolve the ambiguity by choosing the timelike Killing vector that becomes the standard timelike Killing vector $\partial/\partial t$ in a non-rotational frame.

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A. Gauge-fixed equations of motion

In the neighborhood of the conformal boundary it is always possible to write the bulk metric in the form

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j \, , \tag{A.1}$$

where r is a normal coordinate emanating from the boundary and γ_{ij} is the induced metric on the radial hypersurfaces Σ_r. Choosing also the gauge $A_r = 0$ for the gauge field, the gauge-fixed form of the equations of motion is

Einstein:

$$\begin{aligned}
K^2 - K_{ij}K^{ij} &= R + 2\kappa^2 \tilde{T}_{d+1d+1}\,, \\
D_i K^i_j - D_j K &= \kappa^2 \tilde{T}_{jd+1}\,, \\
\dot{K}^i_j + K K^i_j &= R^i_j - \kappa^2 \left(\tilde{T}^i_j + \frac{1}{1-d}\tilde{T}^\sigma_\sigma \delta^i_j \right).
\end{aligned} \tag{A.2}$$

$\dot{K}^i_j$ here stands for $\frac{d}{dr}(\gamma^{ik}K_{kj})$ and $K_{ij} = \frac{1}{2}\dot{\gamma}_{ij}$ is the extrinsic curvature of the hypersurfaces Σ_r. Note also that the components of the Christoffel symbol of the bulk metric are

$$\Gamma^{d+1}_{ij} = -K_{ij}\,, \qquad \Gamma^i_{d+1j} = K^i_j\,, \qquad \Gamma^i_{jk}[g] = \Gamma^i_{jk}[\gamma]\,. \tag{A.3}$$

Vector:

$$\begin{aligned}
D_i(U(\Phi)F^{ri}) &= 0\,, \\
\partial_r(U(\Phi)F^{rj}) + KU(\Phi)F^{rj} + D_i(U(\Phi)F^{ij}) &= 0\,.
\end{aligned} \tag{A.4}$$

Scalar:

$$\begin{aligned}
&\partial_r(G_{IJ}(\Phi)\dot{\Phi}^J) + KG_{IJ}(\Phi)\dot{\Phi}^J + D^i(G_{IJ}(\Phi)\partial_i\Phi^J) - \\
&\quad -\frac{1}{2}\frac{\partial G_{JK}}{\partial \Phi^I}(\dot{\Phi}^J\dot{\Phi}^K + \partial_i\Phi^J\partial^i\Phi^K) - \frac{\partial V}{\partial \Phi^I} - \\
&\quad -\frac{1}{4}\frac{\partial U}{\partial \Phi^I}(2\gamma^{ij}\dot{A}_i\dot{A}_j + F_{ij}F^{ij}) = 0\,.
\end{aligned} \tag{A.5}$$

Here, D_i is the covariant derivative with respect to the induced metric γ_{ij} and $F^r{}_i = \dot{A}_i$ in the gauge $A_r = 0$.

B. Asymptotic CKVs versus asymptotic bulk Killing vectors

We discuss in this appendix the connection between asymptotic bulk isometries and boundary *conformal* isometries. In this discussion we will need a well-known property of the linearized supergravity equations of motion, namely that for each bulk field they admit two linearly independent solutions, the *normalizable* and the *non-normalizable* modes, which near the boundary behave as $e^{-s_+ r}$ and $e^{-s_- r}$ respectively. The exponents s_+, s_- are related to the scaling dimension of the dual operators and the spacetime dimension. Specifically, we have

$$\begin{aligned}
&s^+ = d-2\,, \quad s^- = -2\,, \quad \text{for } \gamma_{ij}\,, \\
&s^+ = d-2\,, \quad s^- = 0\,, \quad \text{for } A_i\,, \\
&s^+ = \Delta_I\,, \quad s^- = d - \Delta_I\,, \quad \text{for } \Phi^I\,,
\end{aligned} \tag{B.1}$$

with $\Delta_I \geq d - \Delta_I$.

Asymptotic conformal Killing vectors.

Definition: we define an asymptotic conformal Killing vector (CKV) to be a bulk vector field ξ which is asymptotically equal to a boundary conformal Killing vector. The precise asymptotic conditions are

$$\begin{aligned} &\text{(i)} \qquad \xi^r = \mathcal{O}(e^{-dr})\,, \\ &\text{(ii)} \qquad \xi^i(x,r) = \zeta^i(x)(1+\mathcal{O}(e^{-(d+2)r})) \end{aligned} \tag{B.2}$$

where $\zeta^i(x)$ is a conformal Killing vector of $\mathrm{g}_{(0)}$.

The asymptotic conformal Killing vectors are in one-to-one correspondence with asymptotic bulk Killing vectors, for if ξ is an asymptotic CKV as defined above, then there exist $\hat{\xi}$, $\hat{\alpha}$, given in (B.7) below, such that $\xi - \hat{\xi}$ is an asymptotic bulk Killing vector, up to a gauge transformation required to preserve the gauge fixing of the vector field, namely

$$\mathcal{L}_{\xi-\hat{\xi}}\psi = \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ r})\,, \tag{B.3}$$

or equivalently

$$\mathcal{L}_{\xi}\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ r})\,. \tag{B.4}$$

To prove this we note that both $\mathcal{L}_{\xi}\psi$ and $\mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi$ satisfy the linearized equations of motion. As noted above, a basis for solutions of the the linearized equations of motion are the normalizable and non-normalizable solution. Since in (B.4) we require equality up to normalizable mode, a sufficient condition for proving (B.4) is that the leading asymptotics between the left and right hand side agree. To show this we note that condition (i) and (3.26)–(3.28) imply that in the gauge (A.1)

$$\mathcal{L}_{\xi}\psi = L_{\xi}\psi + \mathcal{O}(e^{-s_+ r})\,. \tag{B.5}$$

Furthermore, condition (ii) is equivalent to

$$L_{\xi}\psi = \frac{1}{d}D_i\xi^i\delta_D\psi(1+\mathcal{O}(e^{-r}))\,. \tag{B.6}$$

It follows that the leading asymptotics agree with a PBH transformation with parameters,

$$\begin{aligned} \hat{\xi}^r &= \delta\sigma(x)\,, \\ \hat{\xi}^i &= \partial_j\delta\sigma(x)\int_r^\infty dr'\gamma^{ji}(r',x)\,, \\ \hat{\alpha} &= \partial_i\delta\sigma(x)\int_r^\infty dr' A^i(r',x)\,, \end{aligned} \tag{B.7}$$

where

$$\delta\sigma = \frac{1}{d}D_i\xi^i\,, \tag{B.8}$$

which proves our assertion.

Notice that the asymptotic fall-off of ξ^i in (ii) follows from the fact that in order for a vector field to preserve the gauge (A.1) we need

$$\dot{\xi}^i = -\partial^i\xi^r \qquad \Rightarrow \qquad \dot{\xi}^i = \mathcal{O}(e^{-(d+2)r})\,. \tag{B.9}$$

C. Proof of lemma 4.1

In this appendix we give a proof of lemma 1.

C.1 Electric charge

To prove (4.29) we start with the identity

$$\int_{\Sigma_r\cap C} *\mathcal{F} = \int_{\Sigma_r\cap C} d\sigma_i \frac{1}{\sqrt{-\gamma}}\pi^i \,, \tag{C.1}$$

where $\pi^i = -\sqrt{-\gamma}U(\Phi)F^{ri}$ is the gauge field momentum. The second equation in (A.4) can now be written as

$$\dot{\pi}^i = -\partial_j(\sqrt{-\gamma}U(\Phi)F^{ij})\,. \tag{C.2}$$

The momentum π^i and the radial derivative ∂_r can be expanded in eigenfunctions of the dilatation operator as in (3.17) and (3.20) respectively. Moreover, by Taylor expanding $U(\Phi)$ one obtains such an expansion for the r.h.s. of (C.2) too, which takes the form

$$U(\Phi)F^{ij} = \left\{U(0) + \frac{\partial U}{\partial \Phi^I}\Phi^I + \frac{1}{2!}\frac{\partial^2 U}{\partial\Phi^I\partial\Phi^J}\Phi^I\Phi^J + \cdots\right\}F^{ij} \equiv {\varphi_{(4)}}^{ij} + {\varphi_{(5)}}^{ij} + \cdots\,. \tag{C.3}$$

Matching terms of the same dilatation weight on both sides of (C.2) then we obtain

$$\begin{aligned}
{\pi_{(3)}}^i &= 0\,,\\
\sqrt{-\gamma}{\pi_{(4)}}^i &= -\frac{1}{d-4}\partial_j(\sqrt{-\gamma}{\varphi_{(4)}}^{ij})\,,\\
\sqrt{-\gamma}{\pi_{(5)}}^i &= -\frac{1}{d-5}\partial_j\left[\sqrt{-\gamma}{\varphi_{(5)}}^{ij} - \frac{1}{d-4}\delta_{(1)}\left(\sqrt{-\gamma}{\varphi_{(4)}}^{ij}\right)\right],\\
&\;\;\vdots\\
\sqrt{-\gamma}{\tilde{\pi}_{(d)}}^i &= \frac{1}{2}\partial_j\left(\sqrt{-\gamma}{\varphi_{(d)}}^{ij} + \cdots\right).
\end{aligned} \tag{C.4}$$

Therefore, all local terms in the momentum expansion are total derivatives while the non-local term ${\pi_{(d)}}^i$ is left undetermined by this iterative argument. Hence,

$$\int_{\Sigma_{r_o}\cap C} *\mathcal{F} = \int_{\Sigma_{r_o}\cap C} d\sigma_i\frac{1}{\sqrt{-\gamma}}\pi^i = \int_{\Sigma_{r_o}\cap C} d\sigma_i {\pi_{(d)}}^i + \cdots\,. \tag{C.5}$$

Taking the limit $\Sigma_{r_o} \to \partial\mathcal{M}$ then completes the proof of (4.29).

C.2 Charges associated with asymptotic conformal isometries

Applying a similar argument we now prove (4.31). Let, ξ be an asymptotic conformal Killing vector as defined in appendix B, i.e.

$$\mathcal{L}_\xi\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+r})\,, \tag{C.6}$$

where $\hat{\xi}$ and $\hat{\alpha}$, given in (B.7), generate a PBH transformation with conformal factor $\delta\sigma = \frac{1}{d}D_i\xi^i$. Then, using (4.14), (4.17) and the fact that in the gauge (A.1) one has

$$\Xi^{ri} = \nabla^{[r}\xi^{i]} + \kappa^2 U(\Phi)F^{ri}A_j\xi^j$$

$$= \dot{\xi}^i + \Gamma^i_{rj}\xi^j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j \xi^j$$
$$= \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j\right)\xi^j + \mathcal{O}\left(e^{-(d+2)r}\right), \tag{C.7}$$

we can write

$$\int_{\Sigma_{r_o}\cap C}(\mathbf{Q}[\xi] - i_\xi \mathbf{B}) = \frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j\right)\xi^j - \frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \xi^i \left(K_{(d)} + \lambda_{\mathrm{ct}}\right)$$
$$= -\int_{\Sigma_{r_o}\cap C} d\sigma_i \left[\left(2\pi_{(d)}{}^i_j + \pi_{(d)}{}^i A_j\right)\xi^j + \mathcal{O}\left(e^{-(d+2)r}\right)\right] +$$
$$+\frac{1}{\kappa^2}\int_{\Sigma_{r_o}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j - \lambda\delta^i_j\right)_{\mathrm{ct}}\xi^j .$$

Taking the limit $\Sigma_{r_o} \to \partial\mathcal{M}$ we see that (4.31) is equivalent to

$$\int_{\partial\mathcal{M}\cap C} d\sigma_i \left(K^i_j - \frac{\kappa^2}{\sqrt{-\gamma}}\pi^i A_j - \lambda\delta^i_j\right)_{\mathrm{ct}}\xi^j = 0\,, \tag{C.8}$$

which we now prove.

From section 4.1 we know that on-shell

$$\mathrm{d}\mathbf{Q}[\xi] + i_\xi \mathbf{L} = \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi)\,, \tag{C.9}$$

which, using (3.6) and (4.14), can be written as

$$\nabla_\mu \Xi^{\mu\nu} = \kappa^2\xi^\nu\left(-\mathcal{L}_{\mathrm{m}} + \frac{1}{d-1}\tilde{T}^\sigma_\sigma\right) - \kappa^2 v^\nu(\psi, \mathcal{L}_\xi\psi)\,. \tag{C.10}$$

In the gauge (A.1) we can use (3.11) to get

$$\partial_r\left[\sqrt{-\gamma}(\Xi^{ri} - \xi^i\lambda)\right] = \partial_j(\sqrt{-\gamma}\Xi^{ij}) - \kappa^2\sqrt{-\gamma}v^i(\psi, \mathcal{L}_\xi\psi) + \mathcal{O}\left(e^{-2r}\right), \tag{C.11}$$

or, using (C.7),

$$\partial_r\left\{\left[\sqrt{-\gamma}\left(K^i_j - \lambda\delta^i_j\right) - \kappa^2\pi^i A_j\right]\xi^j\right\} = \partial_j(\sqrt{-\gamma}\Xi^{ij}) - \kappa^2\sqrt{-\gamma}v^i(\psi, \mathcal{L}_\xi\psi) + \mathcal{O}\left(e^{-2r}\right). \tag{C.12}$$

To prove (C.8) we only need the time component of this equation. In particular, if $v^t(\psi, \mathcal{L}_\xi\psi) = \mathcal{O}(e^{-(d+2)r})$, then we can expand both sides of (C.12) in eigenfunctions of the dilatation operator using (3.17), as was done for (C.2) in the previous section, and apply the same iterative argument to show that (C.8) holds. Therefore, the proof of (4.31) is complete once we show that $v^t(\psi, \mathcal{L}_\xi\psi) = \mathcal{O}(e^{-(d+2)r})$. As we now explain, this follows from (4.30).

From the explicit form of v^t, given in (3.7), we see that

$$v^t(\psi, \mathcal{L}_\xi\psi) = v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ r})) = v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) + \mathcal{O}\left(e^{-(d+2)r}\right). \tag{C.13}$$

Moreover,

$$\begin{aligned}
v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) \;=\;& -\frac{1}{2\kappa^2}(\gamma^{ti}\gamma^{jk} - \gamma^{tk}\gamma^{ij}) D_k \left(D_{(i}\hat{\xi}_{j)} + 2K_{ij}\delta\sigma \right) + \\
& + U(\Phi) F^{tj} \left(L_{\hat{\xi}} A_j + \dot{A}_j \delta\sigma + \partial_j \hat{\alpha} \right) + G_{IJ}(\Phi)\partial^t \Phi^I \left(\hat{\xi}^i \partial_i \Phi^J + \dot{\Phi}^J \delta\sigma \right) \\
=\;& -\frac{1}{2\kappa^2}(\gamma^{ti}\gamma^{jk} - \gamma^{tk}\gamma^{ij}) \left(D_k D_{(i}\hat{\xi}_{j)} + 2K_{ij} D_k \delta\sigma \right) + \\
& + U(\Phi) F^{tj} \left(L_{\hat{\xi}} A_j + \partial_j \hat{\alpha} \right) + G_{IJ}(\Phi)\partial^t \Phi^I \hat{\xi}^i \partial_i \Phi^J - \\
& - \frac{1}{\kappa^2} \left\{ D^j K^t_j - D^t K - \kappa^2 U(\Phi) F^{tj} \dot{A}_j - \kappa^2 G_{IJ}(\Phi) \partial^t \Phi^I \dot{\Phi}^J \right\} \delta\sigma \, .
\end{aligned} \tag{C.14}$$

The last term inside the braces vanishes by the second equation in (A.2). From (4.30) and (B.7) now follows that $\hat{\alpha} = 0$ and $\hat{\xi}^i$ has no components along the isometry directions. Making repeated use of (4.30) it is then straightforward to show that $v^t(\psi, \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi) = 0$, which completes the proof.

D. Symplectic form on covariant phase space

In this appendix we give the explicit form of the symplectic current on the covariant phase space as given by [47, 23] (see also [24]) and we show that the corresponding pre-symplectic form is well-defined with the boundary conditions (3.44), if there is no anomaly, or (3.51) when the anomaly is non-vanishing.

Symplectic current. The symplectic current $D-1$-form is defined by [23, 24]

$$\omega(\psi, \delta_1\psi, \delta_2\psi) = \delta_2 \mathbf{\Theta}(\psi, \delta_1\psi) - \delta_1 \mathbf{\Theta}(\psi, \delta_2\psi) \, . \tag{D.1}$$

The explicit form of this for the lagrangian (3.1) can be derived directly from (3.7). Writing

$$\omega(\psi, \delta_1\psi, \delta_2\psi) = - * w(\psi, \delta_1\psi, \delta_2\psi) \, , \tag{D.2}$$

with $w^\mu = w^\mu_{\rm gr} + w^\mu_{\rm vec} + w^\mu_{\rm sc}$, we find

$$\begin{aligned}
w^\mu_{\rm gr} =\;& \frac{1}{2\kappa^2} \left(g^{\mu\rho} g^{\nu\kappa} g^{\sigma\lambda} - \frac{1}{2} g^{\mu\nu} g^{\rho\kappa} g^{\sigma\lambda} - \frac{1}{2} g^{\mu\kappa} g^{\nu\lambda} g^{\sigma\rho} - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} g^{\kappa\lambda} + \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} g^{\kappa\lambda} \right) \times \\
& \times (\delta_2 g_{\kappa\lambda} \nabla_\nu \delta_1 g_{\rho\sigma} - \delta_1 g_{\kappa\lambda} \nabla_\nu \delta_2 g_{\rho\sigma}) \, ,
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
w^\mu_{\rm vec} =\;& U(\Phi) \left(\frac{1}{2} g^{\rho\sigma} F^{\mu\nu} - g^{\mu\sigma} F^{\rho\nu} - g^{\nu\sigma} F^{\mu\rho} \right) (\delta_2 g_{\rho\sigma} \delta_1 A_\nu - \delta_1 g_{\rho\sigma} \delta_2 A_\nu) + \\
& + \frac{\partial U(\Phi)}{\partial \Phi^I} F^{\mu\nu} \left(\delta_1 A_\nu \delta_2 \Phi^I - \delta_2 A_\nu \delta_1 \Phi^I \right) + \\
& + U(\Phi) (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) (\delta_1 A_\nu \nabla_\rho \delta_2 A_\sigma - \delta_2 A_\nu \nabla_\rho \delta_1 A_\sigma) \, ,
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
w^\mu_{\rm sc} =\;& G_{IJ}(\Phi) \nabla_\rho \Phi^J \left(\frac{1}{2} g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} \right) (\delta_2 g_{\nu\sigma} \delta_1 \Phi^I - \delta_1 g_{\nu\sigma} \delta_2 \Phi^I) + \\
& + \left(\frac{\partial G_{IJ}(\Phi)}{\partial \Phi^K} - \frac{\partial G_{KJ}(\Phi)}{\partial \Phi^I} \right) \nabla^\mu \Phi^J \delta_1 \Phi^I \delta_2 \Phi^K + \\
& + G_{IJ}(\Phi) (\delta_1 \Phi^I \nabla^\mu \delta_2 \Phi^J - \delta_2 \Phi^I \nabla^\mu \delta_1 \Phi^J) \, .
\end{aligned} \tag{D.5}$$

For the reader's convenience we now compile a list of the most important properties of the symplectic current that we will need, along with the relevant proofs. Further details can be found in [23, 24].

I. If ψ satisfies the equations of motion and $\delta_1\psi$, $\delta_2\psi$ satisfy the linearized equations of motion, then ω is closed

$$\mathrm{d}\omega = 0\,. \tag{D.6}$$

Proof. Taking the second variation of (3.2) and using the fact that the functional derivatives of the lagrangian commute we get

$$\begin{aligned} &\delta_2\delta_1\mathbf{L} = \delta_2\mathbf{E}\delta_1\psi + \mathrm{d}\delta_2\boldsymbol{\Theta}(\psi,\delta_1\psi) = \delta_1\mathbf{E}\delta_2\psi + \mathrm{d}\delta_1\boldsymbol{\Theta}(\psi,\delta_2\psi) = \delta_1\delta_2\mathbf{L} \\ \Rightarrow\quad &\mathrm{d}\omega(\psi,\delta_1\psi,\delta_2\psi) = \delta_1\mathbf{E}\delta_2\psi - \delta_2\mathbf{E}\delta_1\psi\,. \end{aligned} \tag{D.7}$$

This completes the proof since $\delta_1\mathbf{E} = \delta_2\mathbf{E} = 0$, by the hypothesis.

II. For an arbitrary fixed vector field ξ on $\mathcal{M}$ and an arbitrary gauge transformation α, on-shell we have

$$\omega(\psi,\delta\psi,\mathcal{L}_\xi\psi) = \mathrm{d}\left(\delta\mathbf{Q}[\xi] - i_\xi\boldsymbol{\Theta}\right), \tag{D.8}$$

$$\omega(\psi,\delta\psi,\delta_\alpha\psi) = \mathrm{d}\delta\mathbf{Q}_\alpha\,. \tag{D.9}$$

Proof. The variation of the diffeomorphism current with respect to an arbitrary variation $\delta\psi$ of the fields (not necessarily satisfying the linearized equations of motion) is given by

$$\begin{aligned} \delta\mathbf{J}[\xi] &= \delta\boldsymbol{\Theta}(\psi,\mathcal{L}_\xi\psi) - i_\xi\delta\mathbf{L} \\ &= \delta\boldsymbol{\Theta}(\psi,\mathcal{L}_\xi\psi) - i_\xi\mathrm{d}\boldsymbol{\Theta}(\psi,\delta\psi) \\ &= \delta\boldsymbol{\Theta}(\psi,\mathcal{L}_\xi\psi) - \mathcal{L}_\xi\boldsymbol{\Theta}(\psi,\delta\psi) + \mathrm{d}(i_\xi\boldsymbol{\Theta}(\psi,\delta\psi))\,, \end{aligned} \tag{D.10}$$

where the equations of motion, $\mathbf{E} = 0$, have been used together with the identity $\mathcal{L}_\xi = i_\xi\mathrm{d} + \mathrm{d}i_\xi$ on forms. Since $\boldsymbol{\Theta}$ is covariant with respect to bulk diffeomorphisms we have $\mathcal{L}_\xi\boldsymbol{\Theta}(\psi,\delta\psi) = \delta'\boldsymbol{\Theta}(\psi,\delta\psi)$, where $\delta'\psi = \mathcal{L}_\xi\psi$. Hence,

$$\delta\boldsymbol{\Theta}(\psi,\mathcal{L}_\xi\psi) - \mathcal{L}_\xi\boldsymbol{\Theta}(\psi,\delta\psi) = \omega(\psi,\delta\psi,\mathcal{L}_\xi\psi)\,, \tag{D.11}$$

and so

$$\omega(\psi,\delta\psi,\mathcal{L}_\xi\psi) = \delta\mathbf{J}[\xi] - \mathrm{d}(i_\xi\boldsymbol{\Theta})\,. \tag{D.12}$$

Specializing this to solutions, $\delta\psi$, of the linearized equations of motion completes the proof of (D.8).

Moreover,

$$\omega(\psi,\delta\psi,\delta_\alpha\psi) = \delta\boldsymbol{\Theta}(\psi,\delta_\alpha\psi) - \delta_\alpha\boldsymbol{\Theta}(\psi,\delta\psi)\,. \tag{D.13}$$

Gauge invariance implies that the second term on the r.h.s. vanishes and hence, on-shell, we obtain (D.9).

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III. The pullback of the symplectic current on Σ_r takes the form

$$\omega(\psi,\delta_1\psi,\delta_2\psi) = \Big\{\delta_2(\sqrt{-\gamma}\pi_{(d)}{}^{ij})\delta_1\gamma_{ij} + \delta_2(\sqrt{-\gamma}\pi_{(d)}{}^{i})\delta_1 A_i + \\ + \delta_2(\sqrt{-\gamma}\pi_{(\Delta_I)I})\delta_1\Phi^I - 1 \leftrightarrow 2\Big\}d\mu\,. \tag{D.14}$$

Proof. This follows immediately from the form of the pullback (3.42) of $\mathbf{\Theta}$ on Σ_r together with the commutativity of the field variations, $\delta_2\delta_1 - \delta_1\delta_2 = 0$.

Pre-symplectic form. Having established the relevant properties of the symplectic current we now introduce the corresponding pre-symplectic 2-form on the field configuration space. Such a form induces a symplectic form on the solution submanifold of the configuration space [23]. Given a Cauchy surface C, the pre-symplectic form relative to C is defined by [23, 24]

$$\Omega_C(\psi,\delta_1\psi,\delta_2\psi) = \int_C \omega(\psi,\delta_1\psi,\delta_2\psi)\,. \tag{D.15}$$

In order for this to be well-defined obviously the integral on the r.h.s. of (D.15) must converge for all solutions ψ of the field equations and any solutions $\delta_1\psi$, $\delta_2\psi$ of the linearized equations of motion that satisfy the boundary conditions (3.44) - or (3.51) in the case of non-vanishing anomaly. These boundary conditions should also ensure that Ω_C is independent of the Cauchy surface C.

To address these questions we note that the most general solution of the linearized equations of motion satisfying the boundary conditions (3.44) takes the form

$$\delta\psi = \mathcal{L}_{\hat\xi}\psi + \delta_{\hat\alpha}\psi + \hat\delta\psi\,, \tag{D.16}$$

where $\hat\xi$, $\hat\alpha$, given by (B.7), generate a PBH transformation and $\hat\delta\psi = \mathcal{O}(e^{-s^+r})$ is an arbitrary *normalizable* solution. Since, as can be seen from (D.3), (D.4) and (D.5), the pullback of $\omega(\psi,\delta\psi,\hat\delta\psi)$ onto C is $\mathcal{O}(e^{-2r})$, the only contribution to the pre-symplectic form which could be divergent is the integral of $\omega(\psi,\mathcal{L}_{\hat\xi_1}\psi+\delta_{\hat\alpha_1}\psi,\mathcal{L}_{\hat\xi_2}\psi+\delta_{\hat\alpha_2}\psi)$. However, if the background, ψ, satisfies the conditions of lemma 1 and the Weyl factors $\delta\sigma_1$ and $\delta\sigma_2$ are independent of the coordinates adapted to the isometries, then the pullback of $\omega(\psi,\mathcal{L}_{\hat\xi_1}\psi+\delta_{\hat\alpha_1}\psi,\mathcal{L}_{\hat\xi_2}\psi+\delta_{\hat\alpha_2}\psi)$ onto the Cauchy surface C vanishes. Hence, the defining integral (D.15) of Ω_C is convergent.

Next, let C and C' be two Cauchy surfaces bounding a region $\Delta\subset\partial\mathcal{M}$ of the boundary. Using Stokes' theorem and the fact that ω is closed on-shell (property I), we get

$$\int_C \omega(\psi,\delta_1\psi,\delta_2\psi) - \int_{C'}\omega(\psi,\delta_1\psi,\delta_2\psi) = \int_{\Delta\subset\partial\mathcal{M}}\omega(\psi,\delta_1\psi,\delta_2\psi)\,. \tag{D.17}$$

Property III together with the boundary conditions (3.44) and the trace Ward identity (3.22) now give

$$\omega(\psi,\delta_1\psi,\delta_2\psi) = \left\{\delta_2(\sqrt{-\gamma}\mathcal{A})\delta_1\sigma - 1\leftrightarrow 2\right\}d\mu\,. \tag{D.18}$$

Therefore, Ω_C is independent of the Cauchy surface provided we use the boundary conditions (3.44) when the anomaly vanishes, and the boundary conditions (3.51) when there is a non-zero anomaly. This is in perfect agreement with our discussion of the variational problem.

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E. Electric part of the Weyl tensor and the Ashtekar-Magnon mass

In this appendix we briefly discuss the connection between the 'conformal mass' of [1] and our analysis. This issue is also discussed in the the recent work of [9].

The authors of [1, 50] give a definition of the conserved charges for AAdS spacetimes in terms of the electric part of the Weyl tensor, which, in the gauge (A.1), and for arbitrary matter fields, takes the form

$$E^i_j = KK^i_j - K^i_k K^k_j - R^i_j + \frac{\kappa^2}{d-1}\left[(d-2)\tilde{T}^i_j - \left((d-2)\frac{1}{d}\tilde{T}^\sigma_\sigma + \tilde{T}_{d+1d+1}\right)\delta^i_j\right]. \qquad \text{(E.1)}$$

This tensor is traceless due to the Hamilton constraint in (A.2)

$$E^i_i = K^2 - K_{ij}K^{ij} - R - 2\kappa^2\tilde{T}_{d+1d+1} = 0\,. \qquad \text{(E.2)}$$

To make contact with their discussion let us specialize to pure gravity in five dimensions (the inclusion of matter in the discussion is completely straightforward). Expanding this tensor in eigenfunctions of the dilatation operator we immediately see that the term of weight 4 is given by

$$E_{(4)}{}^i_j = 2\left(K_{(4)}{}^i_j - K_{(4)}\delta^i_j\right) + 3K_{(4)}\delta^i_j + K_{(2)}K_{(2)}{}^i_j - K_{(2)}{}^i_k K_{(2)}{}^k_j\,. \qquad \text{(E.3)}$$

Using now the expressions [16, 18, 29]

$$K_{(2)}{}^i_j = \frac{1}{2}\left(R^i_j - \frac{1}{6}R\delta^i_j\right), \qquad K_{(4)} = \frac{1}{24}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right), \qquad \text{(E.4)}$$

we obtain

$$E_{(4)}{}^i_j = -2\kappa^2 T_{(4)}{}^i_j + \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3}RR^i_j + \frac{1}{2}R^k_l R^l_k\delta^i_j - \frac{1}{4}R^2\delta^i_j\right), \qquad \text{(E.5)}$$

where

$$T_{(4)}{}^i_j \equiv -\frac{1}{2\kappa^2}(K_{(4)}{}^i_j - K_{(4)}\delta^i_j) \qquad \text{(E.6)}$$

is the renormalized stress tensor. Therefore, in agreement with Ashtekar and Das [1] and Hollands, Ishibashi and Marolf [9], the difference between the holographic conserved charges, defined using $T_{(4)}{}^i_j$, and the Ashtekar-Magnon charges, defined using $E_{(4)}{}^i_j$, is the tensor

$$H^i_j \equiv \frac{1}{4}\left(-R^i_k R^k_j + \frac{2}{3}RR^i_j + \frac{1}{2}R^k_l R^l_k\delta^i_j - \frac{1}{4}R^2\delta^i_j\right). \qquad \text{(E.7)}$$

As discussed in the main text, this tensor is covariantly conserved and is equal to the holographic stress energy tensor of AdS_5 [18].

There is a similar local tensor that is covariantly conserved when the metric is conformally flat in all even dimensions: it is the holographic stress energy tensor of AdS_{2k+1}. As it was shown in [31], and reviewed in section 2, see (2.7), the Fefferman-Graham expansion of $AdS_{(2k+1)}$ terminates at order z^4 and all terms are locally related to $\mathsf{g}_{(0)}$. It follows that the holographic stress energy tensor, which in general contains the non-local (w.r.t. $\mathsf{g}_{(0)}$) term $\mathsf{g}_{(d)}$, is local in this case. The explicit expression for $d = 6$ is given in (3.21) of [18].

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